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A Tool for Becoming Aware of Attending to Students' Thinking: A Precursor to Developing Teacher Noticing

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ABSTRACT

In this paper, we discuss the development and application of an analytical tool for use with teachers to help them become aware of how they attend to students' thinking during discussions. Through understanding how one explicitly attends to student thinking, corrective measures for improving facilitation of effective classroom discussions can occur. The tool is designed as a decision tree, drawing on literature from both science and mathematics education. The application of the tool in this paper uses video from elementary pre-service teachers' instruction in an early field experience. Utilizing science teaching videos from three teaching teams (Grades K, 2, and 5), we demonstrate how the decision tree tool moves beyond an initial teacher question and student response to bring attention to how pre-service teachers follow up on students' responses. Four main branches are identified (focusing, funneling, acknowledging, and no response), with smaller branches for each. To show the potential of the tool, data is also provided on the frequencies of these decision branches from our analysis of the three teaching teams used to develop the tool. Limitations of the tool, but also implications for future use are discussed.

Keywords: attending to student thinking, professional teacher noticing, pre-service teaching and learning, tool for video analysis

Introduction

Effective science teaching involves teachers understanding how students are making sense of what they are learning (Davis et al., 2020). To do this, teachers must learn to become responsive to their students' needs based on how they are making sense of the phenomenon. A responsive teacher knows how to attend to students' ideas, interpret responses, and respond in a manner that advances student learning (Gotwals & Birmingham, 2016; Kang & Anderson, 2015). These aspects of responsive teaching are strengthened when a teacher is engaged in the practice of "professional noticing" (Sherin et al., 2011; Luna, 2018). Researchers describe the practice of teacher noticing as dynamic and consisting of two activities: (1) giving attention to moments of student thinking in an instructional setting, and (2) making sense of these moments (Luna, 2018). For these two acts to occur, a teacher needs to also consider how they facilitate discussions with students so moments of students' thinking about a particular phenomenon can be brought forward for sense-making (Davis et al., 2020; Windschitl et al., 2018).

To date, much of the research on teacher noticing, and its use in learning to make sense of student thinking, has focused on mathematics education and across a range of contexts – in-service to pre-service and elementary through secondary (see Sherin et al., 2011 for a comprehensive overview). In science education, the concept of teacher noticing has started to gain traction, but it "is still a construct 'under development' without an established definition" (Chan et al., 2021, p. 2). Of the studies in science education that have included specific reference to teacher noticing, more attention is given to studying secondary teachers than primary, but there is a balance in representation between in-service and pre-service teacher studies (Chan et al., 2021). Since Chan et al.'s review, some researchers in science education have started to explore how different practice spaces, such as peer rehearsals (Benedict-Chambers et al., 2020) or online simulations (Lottero-Perdue et al., 2024) can support teacher noticing. Furthermore, there is recent research focusing on secondary science and mathematics pre-service teachers learning to professionally notice as they facilitate discussions within simulated environments and with a focus on argumentation (Zangori et al., 2025).

What is missing within the literature on teacher noticing are analytic tools that can be used with teachers to support attending to students' thinking in the act of facilitating classroom discussion. With the prevalence of approximations of practice (i.e., rehearsals and simulations) in teacher preparation programs, having an analytical tool such as the one described in this paper can provide critical support for pre-service teachers with learning to explicitly identify strategies for interpreting and responding to students and enhance sense-making. Specifically, how are teachers framing questions that follow up a student's response that may help to advance the discussion, redirect the discussion, or, in some cases, unintentionally shut down a discussion. Understanding how to move a discussion beyond the initial question asked is an important precursor for learning how to professionally notice in the act of teaching. The decision tree tool discussed in this paper was designed to meet this need. The tool is easily adaptable in various contexts as it uses video of classroom practice as the medium for analyzing how the teacher is attending to students' thinking in the discussion. In fact, the Zangori et al. (2025) study employed this analytical tool to analyze the secondary mathematics and science pre-service teachers' communication patterns around argumentation. Therefore, the purpose of this paper is to describe the development of the tool and illustrate the possible information that can be gleaned about teachers' attention to students' thinking. To accomplish this goal, 14 teaching videos from three teams of elementary science pre-service teachers' instruction during an early science field experience were used.

Conceptual Framing and Related Literature

Based on the evaluation of a 10-year research-based curriculum improvement project, as well as work with beginning teachers, Davis and Smithey (2009) outline three areas of focus for the development of pre-service elementary science teachers, "a) inquiry-oriented science teaching, b) use of science curriculum materials, and c) anticipating and working with student ideas during instruction" (p. 745). It is the third area that necessitates pre-service teachers to become aware of how they are attending to students' responses during the act of teaching, so they can begin to work towards the more complex practice of professional noticing in the act of teaching.

Professional Noticing

Professional noticing serves as the conceptual framework for the development of the tool described in this paper. Professional noticing, or teacher noticing, first gained acceptance in the field of mathematics education with the work of Sherin and van Es on teachers watching and discussing video of their own classrooms (see Sherin, 2001; 2007; Sherin & van Es, 2005; 2009; van Es & Sherin, 2002; 2006). To notice student thinking involves "(a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom interactions" (van Es & Sherin, 2002, p. 573). For beginning teachers, however, it can be difficult to navigate the many complex skills of professional noticing when there are so many different pedagogical challenges when first learning to teach (Amador et al., 2021; Davis et al., 2006). One of these challenges is learning to be in the moment and listening to what students are saying to decide how to effectively respond. Levin et al. (2009) refer to this act of responding in the moment as attending to students' thinking. This pedagogical move requires teachers to listen to what students are saying in response to questions asked and be able to think in the moment about how to follow up to probe students' thinking further. This skill of knowing how to follow up is important because it is what offers teachers insight into students' thinking and affords them the information needed to learn to develop the more complex noticing skills of 'interpreting' and 'responding' (Jacobs et al., 2010) more effectively. It is important to recognize the various influences that make it challenging for pre-service teachers to develop these complex noticing skills.

Research in science education has increasingly examined how noticing operates in the context of science teaching, where teachers must attend to both general pedagogical cues and the disciplinary dimensions of students' scientific ideas, reasoning, and practices to support their scientific sensemaking (Chan et al., 2021; Russ & Luna, 2013). Science learning is dynamic and context-dependent, requiring teachers to adapt instruction in response to students' evolving ideas (Russ & Luna, 2013). This need for flexibility aligns with research on formative assessment, which underscores that effective noticing involves eliciting, interpreting, and using students' thinking as evidence to make real-time instructional adjustments (Black & Wiliam, 2009).

Noticing in science is influenced by teachers' content knowledge, pedagogical content knowledge (PCK), epistemological framing, and teaching experience (Chan et al., 2021). Luna (2018) emphasized that noticing in science classrooms requires more than identifying correct answers; it demands attention to the disciplinary substance of students' reasoning, which is often implicit and requires interpretation in the moment. Noticing also varies across classroom settings, including whole-class discussions, small-group work, and laboratory investigations (Russ & Luna, 2013).

Despite its importance, novice science teachers often struggle with noticing. Levin et al. (2009) found that beginning teachers can attend to student thinking when supported by environments emphasizing responsive teaching and sensemaking. However, school contexts focused on curriculum coverage and classroom control can undermine these practices. Barnhart and van Es (2015) demonstrated that pre-service secondary science teachers who engaged in video-based analysis developed more sophisticated noticing, particularly in attending to student ideas, analyzing interactions, and proposing responsive instructional moves, than those who did not. However, responding effectively remained the most difficult aspect. Similarly, Luna (2018) observed that elementary science teachers varied in their capacity to notice the disciplinary value in students' ideas, highlighting the need for tools to support pre-service teachers in developing disciplinary noticing skills (Amador et al., 2022).

Although research has established the importance of teacher noticing for eliciting and interpreting student thinking, studies consistently show that responding in-the-moment remains a significant challenge, particularly for pre-service and novice teachers (Barnhart & van Es, 2015; Levin et al., 2009). In science classrooms, this challenge is amplified as teachers must attend to both the substance of students' disciplinary ideas and the epistemic quality of their reasoning while making real-time instructional decisions (Berland et al., 2019; Russ & Luna, 2013). Existing supports, such as video-based reflection and frameworks emphasizing noticing components (Sherin & van Es, 2009; Jacobs et al., 2010), often focus on retrospective analysis rather than immediate instructional guidance. There is a need for tools that bridge the gap between noticing and action during instruction by supporting teachers as they decide how to probe, extend, or scaffold student thinking in real-time. Therefore, this study introduces a decision tree tool designed to fill this gap by providing structured, practical support for pre-service teachers as they respond to students' science ideas during classroom discussions.

Recent work further emphasized the epistemic quality of student ideas as central to noticing in science. Berland et al. (2019) proposed Epistemologically Responsive Science Teaching (ERST), which encourages teachers to notice and respond to the clarity, consistency, and causality (3Cs) within students' scientific thinking. Attending to these epistemic dimensions helps teachers recognize how students' ideas align with scientific sensemaking and supports their engagement in Next Generation Science Standards-aligned practices such as modeling, argumentation, and data analysis. ERST provides pre-service teachers with a coherent structure for lesson planning, real-time noticing, and assessment, helping them to see scientific practices as interconnected rather than isolated.

Further expanding on the relationship between noticing, PCK, and equitable teaching practices, additional studies reveal that novice elementary teachers frequently elicited student thinking but struggled to use that information to guide instruction, especially in science, compared to mathematics (Amador et al., 2022). Benedict-Chambers and Sherwood (2024) similarly emphasized the need for equity-oriented noticing, finding that pre-service teachers often engaged in surface-level descriptive noticing but required support to progress toward evaluative and interpretive noticing, which considers how lesson designs position students as active knowledge constructors.

Within elementary classrooms specifically, much of the interaction around student thinking occurs through verbal exchanges between the teacher and student, or students and students, with the teacher listening (Kelly, 2014). Concerning these patterns in classroom dialogue, Nicol (1999) noted that when pre-service teachers were prompted to examine the questions they asked during their exchanges with students, the pre-service teachers found they asked more yes/no questions rather than probing questions and that their desire to listen for what was expected overshadowed their ability to listen to student reasoning and thinking. Nicol also explained that the preservice teachers recognized deficiencies in both their questioning and listening to student answers and attributed it to their fear of the lesson moving away from their pre-planned instruction. Levin et al. (2009) noted from their study that for pre-service teachers, the issue of learning to attend to students' responses relates to how they are framing their practice. Framing guides what teachers focus their attention on when examining what is happening in the classroom. Unless teachers "frame" their instruction around student thinking, they may not develop the necessary skill of attending to student thinking. Lastly, from their analysis of pre-service teachers' abilities to attend, analyze, and respond to students' thinking, Barnhart and van Es (2015) concluded that for pre-service teachers to provide high levels of analysis and responses to student ideas, they needed to demonstrate sophisticated levels of initially attending to student ideas. Each of these studies demonstrates the importance of first building the skill of learning to attend to students' thinking within discussion and the need to become aware of how they are following up on students' responses to elicit student thinking. For this awareness to develop, beginning teachers, such as pre-service teachers, need to learn about the purpose of questioning and framing of questions.

The Role of Questioning

Certain types of questions can help scaffold students' thinking, assisting them in developing solid conceptual understanding, while others serve only to assess correctness. This first type of questioning is useful because it helps to facilitate learning by giving implicit feedback that further challenges student understanding (Brown & Abell, 2007; Harlen, 2015). It also helps to stimulate more elaboration and productive student responses, leading students to deeper conceptual understandings.

Chin (2006) developed an analytical framework to represent classroom talk and questioning in science to examine how teachers use questioning to engage their students in thinking about content to foster their construction of knowledge. Chin identified various forms of feedback provided by in-service teachers in their follow-up moves during initiation-response-follow-up (IRF) exchange formats (Mehan, 1979). The follow-up moves that generated the most productive student responses were the ones that were non-evaluative and utilized further questioning to elicit deeper thinking (Chin, 2006). Teachers' questions that scaffold students' thinking and lead them to conceptual understanding provide a much greater benefit than those that simply assess correctness.

Another study in mathematics education examined student-teacher interactions in reform-based classrooms where students are encouraged to investigate and share their mathematical thinking (Wood, 1998). In their study, Wood discussed interaction patterns between in-service teachers and students that either encourage or restrict mathematical meaning construction. One specific interaction pattern, initially identified by Bauersfeld (1980), is called the funnel pattern. In the funnel pattern, teachers have a specific answer and/or way of thinking about the content that they are attempting to lead students to state. Asking funneling questions guides students to one determined correct answer rather than encouraging students to share their thoughts about how they are constructing their understanding and application of the mathematical task. Conversely, when teachers follow a different interaction pattern, which Wood (1998) called a focusing pattern, teachers allow students to share their reasoning and thinking without the goal of any specific predetermined answer. Although in both the funneling and the focusing patterns, teachers ask questions of students, only in the focusing pattern are students encouraged to share their strategies and reflect on their mathematical knowledge construction. Focusing questions encourage students to take an active role in making sense of mathematics and remove the imposed limit of only one correct answer. Through asking focusing questions, teachers can examine student thinking and encourage sense-making.

Drawing on this body of literature, we sought to develop an analytical tool to use with video of classroom discussion. We refer to this tool as a decision tree that teacher educators can use with pre-service teachers to develop awareness of how they are initiating dialogue with students (i.e., the initial question) *and* how they follow up the students' responses to promote further discussion and sense-making. The remainder of this paper focuses on describing how the decision tree tool was developed through analysis of multiple video segments of elementary pre-service teachers' early science field experience.

Development of the Analytical Tool

Context and Selection of Videos

The video used for the development of the tool was captured over two semesters during a multi-year NSF-funded Iterative Model Building (IMB) Project (see acknowledgement). The first author in this study was a Co-PI for this project, and the other authors were doctoral students at the time in a class that the first author was teaching. The class involved learning about how to become a science or mathematics teacher educator. The second through fifth authors and the seventh author divided into three teams to review two recorded science lessons taught by three different groups of pre-service teachers. The sixth author in this study provided support with the literature review and conceptual framework for the field of mathematics education.

Taking a case study perspective (Yin, 2009), we selected three teams (cases) of elementary pre-service teachers who had participated in the IMB Project. A total of 14 science lessons across three teams were recorded and all had provided consent for their videos to be used for research purposes. The elementary schools the pre-service teachers taught in for their early field experience were in the same town as the large research university the pre-service teachers attended for their teacher education program. One team (or case) taught in a fifth-grade classroom, another in a second-grade classroom, and the third in a kindergarten classroom. In total, there were 17 females and one male, and 17 of the participants were Caucasian and one was African American. In addition, one of the female participants had selected science as her content area of concentration for her elementary teaching program requirements. This meant that she was required to take one additional science course, a philosophy of science (Nature of Science) course, and one additional science methods course targeting upper elementary/middle grades. However, she did not complete this second methods course until after the semester in which the field experience, where the video for this study was recorded, was taken.

Each week, two to three members of the pre-service teacher teams took the lead in facilitating the science lesson with the class of elementary children. The other team members served as observers or small group assistants. The pre-service teachers leading the instruction of each week's lesson typically split the lesson up into parts, so each person was solely implementing a component of the lesson with the whole class of students. This splitting up of duties allowed for the analysis of one teacher for each segment of the lesson, which helped when identifying the dialogic interactions for the development of the tool, as it was one teacher interacting at a time with the class of students. The videos analyzed included five science lessons on Properties of Fabrics (Kindergarten), five lessons on Properties of Matter (2nd grade), and four lessons on Models and Design using forces on flight for context (5th grade). Each of these topics was requested by the corresponding classroom mentor teacher to align with the school district's adopted science curriculum.

Phase One – Beta Testing Codes

Given the video we were using to develop the tool was of pre-service elementary teachers' science teaching, our first phase of coding was informed by Levin et al. 's (2009) definition of attending to student thinking, which they described as a teacher "notic[ing] and respond[ing] to a student's idea" (p. 147). They further explained different ways teachers can respond including: 1) asking a student or other students to explain or elaborate on an idea, 2) rephrasing what the student shared, and 3) shifting the lesson to address the idea shared (Levin et. al., 2009). We utilized these descriptions initially as the codes for understanding how the pre-service teachers were attending to student thinking in the act of teaching. In addition to the three attending codes, we also developed the code 'acknowledging' to document the ways that pre-service teachers were not attending to student thinking in their instruction. 'Acknowledging' meant that the pre-service teacher gave little to no consideration as to what the student said before returning a response. Therefore, the teacher's response for an acknowledging code included providing general praise, evaluating the answer, or asking the student to repeat the response. See Table 1 for a summary of the four levels of coding and their descriptions for the first round of a priori coding used in the development of the tool.

Table 1

Initial Draft of Codes Developed for Analytical Tool Adapted from Levin et al., 2009

Code	Definition
Acknowledging	The teacher did not consider what the student said before returning a response that suggested they were not fully listening.
Attending - shift	The teacher hears the topic and makes a related comment in return, but then shifts or pivots to a new topic with the next question.
Attending - rephrasing	The teacher takes what the student said and rephrases it to help others understand or asks the student who shared if the rephrased statement accurately represents what they were saying.
Attending - elaborate	The teacher shifts the attention to a student peer to repeat or elaborate on the statement the initial student made.

To begin with the a priori coding noted in Table 1, each pair of coders prepared transcripts from watching the videos for the first two lessons taught by the team of pre-service teachers they were assigned to. Using these transcripts, Authors 2-5 and 7 coding partners first met to identify episodes in the transcript demonstrating an initiation- response- follow-up (IRF) exchange (Kelly, 2014). Identifying these exchanges was important to give boundaries for coding and to ensure each episode included a teacher initiating with a question, a student responding, and then the teacher following up. It is how the teacher responds to the follow-up segment and utilizes the conversation for instruction that the decision tree tool serves its purpose. The ending of an episode was identified as ending when no further or substantial ideas were being shared by students, or a new topic was identified. Therefore, some episodes are longer than others, and episodes could be between the teacher and one student or the teacher and several students. We did not move to the next level of analysis until the team was in 100% agreement as to this definition of identifying an episode, and each coding pair agreed on the identification of episodes, using this definition, for their assigned videos/transcripts.

The next step involved the coding partners 1) individually coding a whole transcript for lesson one, then 2) coming together and reconciling their results. This was repeated for lesson two. A coder first needed to identify within each identified episode if the pre-service teacher leading the discussion indeed was showing some attention to student thinking in their follow-up response to the student's comment. This became the first level of the decision tree: Does the teacher attend to the student's thinking? Referring to codes generated from Levin and colleagues, this first level of coding determined whether the coders would move down the path of yes – and possibly one of the three 'attending' codes described in Table 1, or no, move to the acknowledging code instead. Codes associated with acknowledging follow-up emerged through this process.

During this beta-testing phase of coding, the coders were asked to also highlight any exchanges that they did not feel aligned with codes and/or subcodes on our initial coding structure

(Table 1). When Author one and the three sets of partner coders came together to reconcile their codes for the beta-testing phase, they discussed any of these highlighted exchanges until 100 % agreement was reached. From these discussions, we noticed there were more nuanced examples of how the pre-service teachers were attending to student thinking, and not attending to their thinking, than our initial coding tool (Table 1) allowed for. Therefore, to better understand how a teacher considers a student's ideas before responding, we further explored the research base and found two variations in the mathematics education literature on communication patterns that captured the distinct differences in 'attending to student thinking' we identified from the first round of analysis. These variations are described further in the next section as we explain refining the coding process.

Phase Two - Refining the Coding Schema

It was through this refining process that we began to identify the tool as a decision tree because more than one decision needed to be made beyond the initial question of whether the pre-service teacher was attending to students' thinking. Through this refinement process, additional sub-codes (or smaller branches of the tree were identified).

To begin the refinement process, the research team drew on the work of Wood (1998) in mathematics education that discusses communication patterns in the classroom. The first variation of attending to students' thinking we adopted from Wood was the notion of 'funneling', which refers to a teacher using well-intentioned questions to guide or focus students' thinking to a particular outcome. The second variation of attending to students' thinking was the concept of 'focusing', which involves the teacher creating situations for classroom talk that allows students to explain and give reasons for their science ideas. To further expand on each of these codes, the researchers returned to the previously coded transcripts for the first two videos of lessons taught by each pre-service teaching team. Again, coding partners individually reviewed the identified bounded episodes using the new three-level coding schema of focusing, funneling, and acknowledging. Within each of these coded segments, which we refer to in the decision tree as a branch, additional emergent coding occurred to identify different examples of focusing, funneling, and acknowledging. These examples became the smaller branch decisions in the tool. Allowing for the emergent coding also afforded the opportunity to look for various ways that attending to students' thinking and for different purposes can occur. This process also led to identifying a second larger branch of not attending to student thinking, which we called 'no response'. This non-attending branch, just as the title suggests, was coded in contrast to acknowledging and examples of it occurred when the pre-service teacher made follow-up comments that showed no consideration of the student's comment. This could include affirmatory or ignoring the comment, and otherwise, but it often resulted in shutting down the discussion or completely changing the direction of the discussion.

As coding partners came together, and then the larger research team, to discuss these examples, we were able to condense emergent codes across all four main branches (funneling, focusing, acknowledging, or no response), provide definitions for each, as well as the smaller branches coming from each. These definitions and examples of each from episodes identified in the data were reached with 100% agreement among the research team and are provided in Appendix A.

The recursive process the research team used to generate this code book for the decision tree helps to ensure inter-rater reliability and validity in the coding process and applicability of the tool across grades and topics. Because of the tiered analysis process, specific inter-rater reliability calculations were not made, as the goal was to reach agreement across all of our identified episodes to produce a comprehensive tool. For the final step, and to determine saturation in the definitions for each branch of the tool, the coding partners returned to the final three lessons taught by each pre-service teaching team and applied the decision tree tool to ensure no additional codes emerged.

Codes identified in the tree represented each of the follow-up responses a teacher provided, thus a point of saturation in coding was reached.

Final Steps of Development and Application

In the end, three smaller branches were identified for the main 'focusing' branch, four smaller branches were identified for each of the funneling and acknowledging main branches, and the fourth main branch, 'no response', also had three smaller branches stemming from it. Each main branch and the smaller branches extending from them are described below.

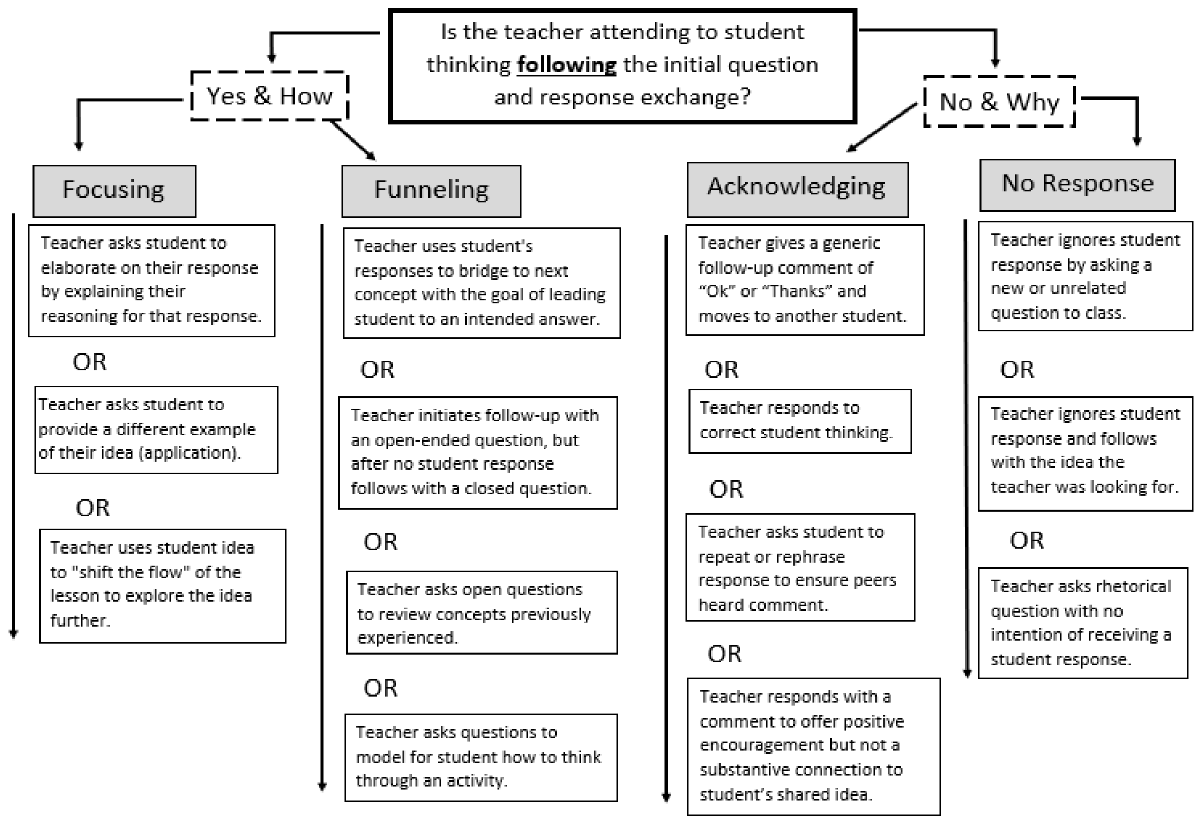
1. *Focused, attending*: When pre-service teachers were focused on what the students were saying they would: 1) ask students to elaborate on their response and explain their reasoning, 2) ask students to provide an application of their idea, or 3) shift the flow of the lesson to explore a student's idea further.
2. *Funneled, attending*: When the pre-service teachers funneled students' ideas towards a specific learning outcome they would: 1) use the student responses to bridge to the next concept to lead students to the intended answer; 2) ask an open-ended question, but after no response follow-up with a closed question; 3) ask an open-ended question to review past concepts; 4) ask questions that are intended to model for students how they should be thinking through an activity.
3. *Acknowledging, not attending*: For this branch it was observed that the pre-service teachers responded to a student's comment by: 1) recognizing the student said something but not fully paying attention to what was said because their next follow-up comment was unclear, unrelated, or referenced only a part of the student's idea; 2) paying attention to what was said but for correctness; 3) paying attention to what was said but only to motivate or encourage the student to participate; 4) asking students to repeat or rephrase their responses to ensure it was heard by others but no connection was made to the idea the student shared.
4. *No response, not attending*: This branch included examples of when the pre-service teacher 1) clearly ignored a student's response by asking a new and unrelated question, 2) stated the idea to the students' they were looking for as a student response; 3) asked a rhetorical question as a follow-up to the student's response. Although this type of exchange suggests limited communication, we believe it is important to include this code, as it affords teachers the ability to recognize *how* teacher responses to a student comment can also cause communication patterns to be limited or even stopped.

Figure 1 illustrates the decision-tree tool, showing these four main branches coming from the initial question: *Is the teacher attending to students' thinking in the discussion?* If the answer to this question is yes, then the teacher follows the left path on the decision tree to code for how. If the teacher is not attending to students' thinking, then follow the right path of the decision tree to identify why. We recommend this tool be used with video, and preferably transcripts of the video, if available. Having both the video of instruction and the transcripts can make it easier for first bounding episodes for coding. However, it could potentially be used as an in-the-moment observation tool, but with space provided for documenting frequency counts and perhaps a section for notes to summarize an example of one exchange to illustrate a decision most frequently made by a teacher in discussion with students.

We also recognize the smaller branches under each of the four main codes are not an exhaustive list of possible responses a teacher may give in following-up from a student's comment, but these branches (or examples) were reached through an iterative process of reaching saturation across the 14 videos for the data set we had access to for this study. Therefore, depending on whether the tool is used with pre-service teacher video or exemplary teacher video, it is possible that additional smaller branches of interactions could be found. The design of this tool allows for this kind of flexibility.

Figure 1

Decision Tree Tool for Developing Awareness About Attending to Students' Thinking During Discussions



What We Learned About Pre-service Teachers' Attention to Students' Thinking

To illustrate the potential of information that can be gleaned from using the tool with pre-service teachers' own teaching video, we kept records of our coding results across the three teams of pre-service teachers. The purpose of this section is to share these results to illustrate the potential use and outcomes of the tool.

In total, there were 291 coded Initiate-Respond-Follow-up (IRF) segments, or what we refer to as the bounded episodes of classroom dialogue. This number of episodes includes the coding across all three teaching teams (cases) for the full set of 14 lessons. Of the 291 episodes, the kindergarten case had 101 coded episodes in five lessons, the second grade case had 110 coded episodes in five lessons, and the fifth grade case had 80 coded episodes across their four lessons. Table 2 provides a breakdown of how these episodes were identified across the four main branches of focusing, funneling, acknowledging, and no response, and the smaller branches for each case.

Table 2

Frequencies by All Branch Levels Compared Across All Three Teaching Teams

Decision Tree Branches – Main and Smaller Branches	Grade K (*n*=101)	Grade 2 (*n*=110)	Grade 5 (*n*=80)
No response			
Teacher ignores student response by asking a new or unrelated question to class.	5	2	3
Teacher ignores student response and follows with the idea the teacher was looking for	8	3	3
Teacher asks rhetorical question with no intention of receiving a student response.	4	4	0
Totals	17	9	6
Acknowledging			
Teacher gives a generic follow-up comment of "Ok" or "Thanks" and moves to another student.	6	9	2
Teacher responds to correct student thinking	15	39	16
Teacher asks student to repeat or rephrase response to ensure peers heard comment.	3	3	3
Teacher responds with a comment to offer positive encouragement but not a substantive connection to student's shared idea.	18	30	12
Totals	42	81	33
Funneling			
Teacher uses student's responses to bridge to next concept with the goal of leading student to an intended answer.	16	10	25
Teacher initiates follow-up with an open-ended question, but after no student response follows with a closed question.	5	7	2
Teacher asks open questions to review concepts previously experienced.	7	2	2
Teacher asks questions to model for student how to think through an activity.	6	0	0
Totals	34	19	29
Focusing			
Teacher asks student to elaborate on their response by explaining their reasoning for that response.	7	0	11
Teacher asks student to provide a different example of their idea (application).	0	1	1
Teacher uses student idea to "shift the flow" of the lesson to explore the idea further.	1	0	0
Totals	8	1	12

Of the four main branches, we found that most of the episodes (156 of 291 total or 53.6%) were coded to acknowledging. More specifically, the second grade case had 81 of their 110 (73.6%) coded as acknowledging; whereas the kindergarten case had 42 of their 101 coded episodes (41.6%), and the fifth grade case had 33 of their 80 episodes (41.3%) coded as acknowledging. Examining this difference between the second grade case and the other two cases further, we see the second grade case had many more episodes coded for responding to correct thinking by students or offering positive encouragement of a response, despite the quality of what the student said. This suggests the pre-service teachers on the second grade team, compared to the other two cases, either viewed their purpose for questioning to ensure accurate answers were shared, or that they struggled with asking questions that went deeper in understanding students' thinking. Becoming aware of and understanding the reasons for these patterns can provide critical information for teacher educators, or even a teacher themselves, to try and plan for better follow-up responses that will extend the conversations further and build more or connect more with different students' ideas.

The second main branch in total frequencies was attending to branch of funneling. Overall, this branch received 28.2% (82 of the total 291) of the coded episodes. Specifically, the kindergarten case had 34 of 101 coded episodes (33.7%) to this category, second grade had 19 of 110 coded segments (17.3%), and fifth grade had the highest frequency with 29 of 80 coded episodes (36.3%) as funneling. Of the smaller branches coming from funneling, the one receiving the most coded episodes was – *Teacher uses student's responses to bridge to next concept with the goal of leading student to an intended answer*. This smaller branch received 51 of the total 82 (62.2%) coded episodes for funneling. These findings suggest that the pre-service teachers across all three cases valued hearing students' ideas, and to bridge from one student's idea to the next concept, and often to lead the conversation to a specific goal. Being aware of this communication pattern offers teacher educators and teachers opportunities to discuss alternative questioning techniques with teachers that could foster more sense-making opportunities to build students' ideas collectively and press for more evidence-based explanations (Windschitl et al., 2018). Identifying these patterns can also open up the opportunity to show teachers how to use more productive wait time before responding to a student's comment.

The main branch with the fewest total coded episodes was focusing. The smaller branches associated with this main branch also demonstrate the most connection to students' thinking is being attended to, in such a way that it is driving the communication pattern. Overall, fifth grade had 12 of their 80 episodes (15%) coded to the focusing branch, kindergarten had 8 of their 101 coded episodes (7.9%) coded to this branch, and second grade had 1 of their 110 episodes coded to this branch (0.9%). Becoming aware that the attending branch of focusing is the branch that teachers are least demonstrating an ability to support as a communication pattern is telling for teacher educators. This branch is the one that is most likely to support students to engage in the discussion for sense-making purposes and build reasoning about the science concepts by connecting the students' ideas (Davis et al., 2020). Thus, more attention needs to be given in teacher education programs on how to help novice teachers navigate these complex patterns of classroom talk (Abell et al., 2010; Davis et al., 2020; Gallas, 1995).

These results illustrate the potential for information that can be produced using the decision tree tool. This kind of information can provide not only teacher educators with a broad view of how their pre-service teachers are responding to student thinking but also bring awareness to pre-service teachers themselves if they were to use this tool to analyze their own teaching video. Reflecting on this information about how, and why, a teacher is attending to (or not attending to) students' thinking in the act of teaching can help to develop awareness of what are both the strengths and weaknesses in one's practice (Gotwals & Birmingham, 2016). Additionally, it can assist with understanding questions that work well to initiate discussions and welcome student comments that can then be attended to more easily (Elstgeest, 2001). In turn, this can lead to more productive classroom discussions (Gallas, 1995). In conclusion, understanding this more detailed level of analysis, which the decision tree tool offers, affords teacher educators a starting place to talk with their preservice teachers about *attending* to students' ideas as the critical first step or precursor to professional noticing.

Limitations and Next Steps

Although this tool has the potential to significantly contribute to the field of teacher education, we recognize that the process used to develop the tool has some limitations. First, we developed this only using elementary pre-service teacher videos. To determine if it is applicable for use with middle or secondary contexts, the tool would need to be tested with video in those contexts. Recently, the first author published a piece with a collection of secondary mathematics and science teacher educators (Zangori et al., 2025). Second, with the focus on pre-service teachers practice as the context, it is uncertain if all these smaller branches would also show for in-service teachers, especially exemplary classroom teachers. Again, the tool should be applied to videos of these teachers' practice determining its utility for these contexts. Given the initial purpose of this tool was to provide teacher educators and/or their pre-service teachers with a user-friendly tool for analyzing practice and to become aware of their attention to (or no attention to) students' thinking in their novice teaching contexts, the data reported from the application of the tool with the three teams of pre-service teachers illustrates the tool meets this purpose.

Furthermore, our tool was designed primarily to identify pre-service teachers' attention to student thinking. However, recent research highlights that noticing in science classrooms must also account for the epistemic and equity dimensions of student sensemaking (Berland et al., 2019; Benedict-Chambers & Sherwood, 2024; Rosebery et al., 2016). Future adaptations of the tool could incorporate these dimensions to better capture the complexity of science classroom interactions and support teachers in making responsive and equity-oriented instructional moves.

Regarding future use for the decision tree tool, using it to compare different levels of expertise in teaching science could be a productive approach to developing pre-service teachers' ideas about how to hold science talks (Gallas, 1995). Many elementary teachers' own science learning experiences perhaps followed more traditional approaches with less emphasis on understanding scientific reasoning, and thus their knowledge of how to talk about science ideas may be limited (Appleton & Kindt, 2002). Holding these discussions in the context of their pre-service science methods class may help to develop their identities as teachers of science because they are engaged in the act of professional noticing with peers (Abell et al., 2010; Davis & Smithey, 2009; Jacobs et al., 2010; Sherin & van Es, 2009).

Additionally, longitudinal research could trace pre-service teachers' noticing development from their initial teacher education into their early years in the classroom, addressing calls for such studies (Barnhart & van Es, 2015; Chan et al., 2021). Such research could illuminate whether tools like this support sustained growth in teachers' capacity to notice and respond to students' ideas as they transition into in-service teaching. Furthermore, examining how pre-service and early career teachers' epistemological framing of their noticing practices could provide deeper insight into how noticing tools interact with teachers' beliefs (Luna, 2018; Russ & Luna, 2013).

Finally, in addition to its use with pre-service teachers, we believe this tool has great potential for supporting in-service teacher development. For many of the same reasons this tool could benefit pre-service teachers in developing their awareness of how they are attending to student thinking, this tool could also assist in-service teachers with learning how they facilitate classroom interactions to promote students' scientific thinking (Sherin & van Es, 2005; Talanquer et al., 2013). For example, teachers could use this tool within the context of professional learning communities to support one another by identifying patterns in their classroom conversations with students (Lave, 1991). For professional developers working with in-service teachers, this tool could help them initially identify individual teacher needs to target in their professional development projects. This level of identification would enable professional developers to track changes in teachers' practice throughout the project and determine if teachers are meeting the professional development goals.

Moreover, integrating the tool into professional development settings where teachers collaboratively analyze video records of their own and others' classrooms could enhance its impact (Barnhart & van Es, 2015; Sherin & van Es, 2009). Research suggests that video-based professional development fosters teachers' ability to attend to the substance of student thinking and reflect on their own practice (Amador et al., 2022; Sherin & van Es, 2005). Professional development using the tool could also emphasize the importance of equity-focused noticing, encouraging teachers to consider how their instructional moves position students as capable contributors to scientific discourse (Benedict-Chambers & Sherwood, 2024; Rosebery et al., 2016).

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Appendix

Examples of Episodes Used with the Development of Decision Tree Main and Smaller Branches.

Branches	Dialogue Examples
FOCUSING	
Teacher asks student to elaborate on their response by explaining their reasoning for that response.	T1: …Did anybody else try using paper besides lined? S: We tried to use tissue, but it didn't work. T2: It didn't work? Why don't you think it worked? S: Because it was way too thin.
Teacher asks student to provide a different example of their idea (application).	T1: Can somebody give an example of what it means when something is two dimensions or something is three dimensions? S1: 2-D is flat and three D actually like pops out. T1: Okay T2: Does that make sense? Like somebody give me an example of something that's 3-D? S2: Ms. Young's desk. T1: What about 2-D? S1: Paper
Teacher uses student idea to "shift the flow" of the lesson to explore the idea further.	S1: Um, some are different and not all the same color. T: And that's OK because right here we are sorting them by size. But could you sort them by color too? S1: *Nods to indicate yes.* T: You could? How would you do that?

Branches	Dialogue Examples
	S1: You would see if they are the same color. T: OK, so maybe if you had three oranges ones you'd put them here. S1: *Nods to indicate yes.* T: If you had two green ones you would put them in a different pile.
FUNNELING	
Teacher uses student's responses to bridge to next concept with the goal of leading student to an intended answer	T: An ice cube. So we'd have ice cubes in here, they would be a solid. What did we just do? We changed a liquid into a… S1: Solid? T: Solid. OK, and then if I'm like, well, I'm tired of these ice cubes…I'm going to dump them out, we're all gonna go to recess, then when we come back…what would happen to our ice cubes? S1: They would turn into water. T: They would turn in to water. Which one did we say water was? S1: A liquid. T: A liquid. So, we would have essentially a puddle of water, right? (Doesn't wait for student responses.) OK…so our state changes that we're talking about, they are from a liquid to a solid and a solid to a liquid. So, I just gave you the example of turning water into ice and then turning the ice back into water.
Teacher initiates follow-up with an open-ended question, but after no student response follows with a closed question.	T: Okay, now let's talk about item A. What did you guys notice about that? S1: It is a lotion. T: How did you know it was a lotion? S1: It smells like it and feels like it. T: It smells like lotion and feels like lotion? Okay, We will now talk about matter. What is a lotion? S2: It is solid. T: She thinks it is a solid. What is everybody else thinking? S3: Liquid T: He thinks it is liquid. Why do you think it is liquid? S3: It moves. T: Alex says lotion is movable. That is very good. What else about lotion makes it liquid or solid? If you think it is a liquid raise your hands. Do you think it is a solid?
Teacher asks open questions to review concepts previously experienced.	T: OK boys and girls, who can tell me what we worked with last week? S1 : Fabric. T: Fabric. What did we use with the fabric? S2: Um, we heard the sounds we hear. T: The sounds with fabric. What else? S3: We used our five senses. T: Our five senses, good. Can you guys remind me what the five senses are? S4: We didn't use one of them. T: Which one didn't we use? S4: Eat. T: Eat. Our sense of taste *(points to her mouth)*. We don't want to taste the fabric. S (all): *Laughing and making yuck sounds.*

Branches	**Dialogue Examples**
	T: So which ones did you use? S5: Uh…see, uh hearing. T: Our hearing, OK. S6: Our sense of touch. T: Our sense of touch. Our sense of hearing. [*Holds up 2 fingers to keep track*]. S7: Our sense of smell. T: Smell and S8: Our sense of see. T: Sight. OK, that's four. And we didn't use taste, right? S2: No.
Teacher asks questions to model for student how to think through an activity.	T1: So who would like to help me sort these into two piles? Do you want to come up here. How could we sort all these fabrics into two piles? T2: What's one sense that you could use to sort them? S1: Feel T1: Okay, how does it feel – if you look at the word wall – soft, smooth, bumpy, hard? S1: It feels bumpy. T1: Bumpy. Okay do any of the others feel bumpy? Does that one feel bumpy? S1: Nods head T1: Any others? S1: Student shakes head. T1: No okay so we can sort these into one pile. Does someone want to come up and make a second pile? T2: So instead of bumpy what kind of feeling do you want to use? S2: These are soft. T1: So these three feel soft.
ACKNOWLEDGING	
Teacher gives a generic follow-up comment of "Ok" or "Thanks" and moves to another student.	T: Okay can anybody, just in case anybody else in the class might not know, can anybody tell me what it means to make a prediction? Does anybody remember what a prediction is? S: Umm it's when you make a prediction it's not a right or wrong answer. T: Okay so prediction is when you think prior to actually acting out your investigation. So last week we actually made predictions about what would happen if water was put onto those fabrics and some people said they were going to stay on top, some said they were going to go through, but when we actually went to the tables and did the activity we were able to come up to the front and fill out our chart and put what happened when the water was put on the fabrics. And this week we are going to do the same thing.
Teacher responds to correct student thinking.	T: In your journals, can you write something else that you think has gas in it? S: Basketball T: A basketball. That is right. It has gas in it.

Branches	Dialogue Examples
Teacher asks student to repeat or rephrase response to ensure peers heard comment.	T: What do we hear? What are some things that we can hear? S2: Umm pop stuff. T: (leans forward to student) You can hear what? S2: (repeats) Pop stuff T: Things that pop. Yeah, things that might pop.
Teacher responds with a comment to offer positive encouragement but not a substantive connection to student's shared idea.	T: Yeah did you have one to share? S: Yep. T: Okay can you tell us about it? S: This one has colors all over the place. T: So this one has lots of colors. S: And then light ones. T: So these are light colors for his second pile. And what's your third one? S: Dark colors T: And this one was a dark color.
NO RESPONSE	
Teacher ignores student response by asking a new or unrelated question to class.	T: Let's say my Dad is architect and needs to build a structure between 2 cities, but there is a river between them. Is there any type of structure that you think would work well? S1: An arc? T: An arc. Ok. What would an arc do? S2: It's just like a little rainbow T: Now I want cars to be able to travel from city to city
Teacher ignores student response and follows with the idea the teacher was looking for.	T: We already used sight and we already used feel so what sense do we still need to use? S: I'm not going to taste. T: So we didn't use our hearing. S1: Got it. T: Okay how did you do it? S: Explaining (can't hear). T: Okay but shat sense have we used? We've used touch, we used sight, so let's try to listen. What did you say? What did you say this sounded like? S1: It sounds like sand. T: So could we put all fabrics that sound like sand together. S1: None of them sound like sand. T: Can we put them in three groups? S1: Explains her three piles.
Teacher asks rhetorical question with no intention of receiving a student response.	T: Ok, if you could- you can't really see DNA with your hands, like if you held it out in front of you, it would be really tiny, you couldn't see it. Do you think it would be easy to learn from that if you can't see it? Like, the real-life thing? It would be hard to learn from.

Note: Each row is a separate episode coded from across all 15 videos. S = student; T = Teacher. Students and teachers change from row to row, providing representation of all three grades and pre-service teaching teams.

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Effectiveness of Problem-Centered Learning in Enhancing Senior High School Students' Achievements in Genetics

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ABSTRACT

The 21st-century skills seek to develop learners who will be problem solvers in society. This paradigm shift requires the use of learner-centered strategies that emphasize collaborative problem-solving. To explore the effectiveness of such approaches, this study sought to determine how effective Wheatley's problem-centered approach would be in the attainment of learning outcomes of students in genetics by comparing it to the conventional teaching approach. Using an embedded mixed-methods design, an intact class was selected randomly from two schools. The pre-test-post-test non-equivalent group design was used to obtain quantitative data, while interviews were used to obtain qualitative data. The students in the experimental group learned biology concepts using the Wheatley model while those in the control group were taught the same concepts through the conventional teaching approach. Students in the experimental group performed better on the post-test than those in the control group. The performance of the low achievers within the experimental group also improved. Students expressed an overall positive attitude toward the use of the Wheatley model as an instructional strategy. It was therefore recommended that biology teachers should employ the Wheatley model in the teaching and learning of biology at the Senior High School level.

Keywords: problem-centered learning; Wheatley model; high school biology; instructional approach; constructivism

Introduction

Stakeholders in education over the years have placed responsibility on teachers for students' academic failures. Such blame emanates from the fact that teachers are seen as an integral component of the schooling process. Teachers' personalities, knowledge, attitudes, and pedagogical strategies affect the attainment of learning outcomes for students. For successful and effective teaching, teachers are required to select appropriate pedagogical strategies that can maximize students' learning. Abell et al. (2010) argued that teachers' pedagogical actions affect students' learning outcomes. Therefore, poor teaching approaches are claimed to be one of the major pivotal issues to students' poor performance in the sciences (Abell et al., 2010; Hassard & Dias, 2013).

Consequently, teachers must explore and use appropriate approaches to teaching to facilitate and maximize students' learning. However, the task of selecting an appropriate pedagogical strategy becomes daunting with the availability of several teaching methods. Although there are several teaching approaches, educationists believe that for effective teaching, teachers must employ constructivist teaching strategies (Fosnot, 1989; Steffe & Gale, 1995; Tobin & Tippins, 2012; Zemelman et al., 1993).

The call for the utilization of constructivist teaching approaches is hinged on the assumption that such approaches are student-centered, foster student collaboration, and increase students' academic achievement. The learners are assumed to create their knowledge and meaning through interaction with others (Hand et al., 1997). Learners, through their personal experiences, socially interact with others to arrive at an appropriate understanding of content (Bruning et al., 2004). Individual learners ultimately construct a personal version of the socially negotiated meaning (Taber, 2012). Learners' conceptualizations of scientific concepts are enhanced as they interact, discuss, and elaborate ideas with their colleagues (Mazur, 1997).

Aulls (2002) refers to such discussion as academic discourse and notes that substantive academic discourse facilitates students' exploration of curriculum topics and material. Students' ability to argue and communicate is enhanced through discussions they engage in during cooperative and group learning (Pagan, 2016). Such discussion involves students talking about the subject and arriving at their conclusions rather than simple, routine interactions between instructors and students (Aulls, 2002). Students' learning, therefore, is not based on the instructor's instruction but rather on the students' work (Schuh, 2003).

Constructivist approaches provide students with skills that can be used outside the classroom as well as reinforce social cognition (Aulls, 2002). These traits are developed because constructivist instructional goals facilitate student application of external knowledge sources and encourage students to utilize scientific reasoning as they solve realistic, real-life problems collaboratively, leading to the development of elicited social cognition (Echevarria, 2003; Petraglia, 1998). Thus, effective constructivist instruction leads to the development of various skills needed for a successful existence in the 21st century.

The developers of the Ghanaian Senior High School (SHS) Biology syllabus realize the strengths of student-centered approaches and have therefore advocated for its use in the teaching and learning of Biology concepts at the SHS level (Curriculum Research and Development Division [CRDD], 2010). Moreover, the new Ghanaian curriculum for the pre-tertiary level also emphasizes that science education should produce learners who will be problem solvers and innovators through the contextualisation of learning, making students act as knowledge creators (National Council for Curriculum and Assessment [NaCCA], 2020). The idea is to enable learners to become knowledge creators, drawing from their prior experiences. This shift in teaching philosophy calls for constructivist paradigms and approaches.

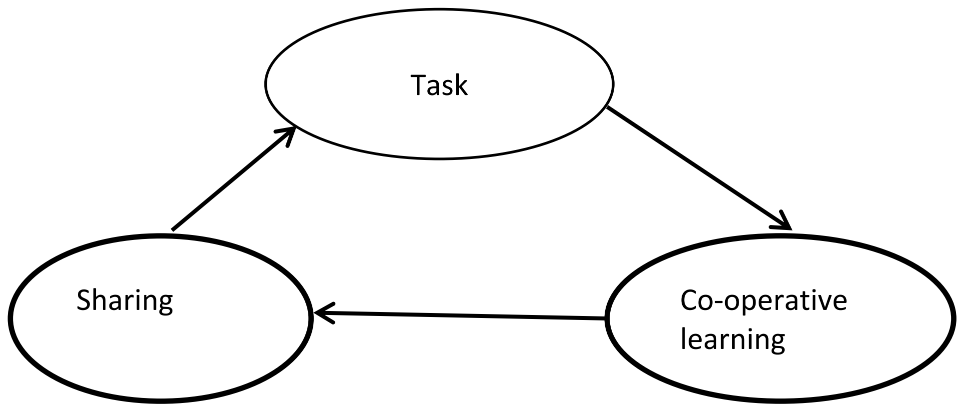
Unfortunately, the call for a shift to constructivist approaches is not supported by the suggestion of specific strategies to be used even though Ghana operates a syllabus system where teaching activities are explicitly suggested. This creates a situation where Ghanaian biology teachers find themselves in a conundrum as to which of the approaches will be best suited for the peculiarities of their students' learning and their classroom environments. Thus, to successfully entice teachers to use constructivist approaches, efforts should be made to identify strategies that will be appropriate for the Ghanaian context.

One of the constructivist approaches that is effective and efficiently employs real-life contextualized problems is the Wheatley model (Wheatley, 1989). Wheatley pointed out that knowledge is not passively received but actively built up by the student in the learning process through the solution of realistic problems in a social setting. In this approach, a student's ability to organize information in unique ways and relate with others to have shared knowledge in a community constitutes learning. To facilitate the development of personalized learning, teachers are expected to guide learners through the provision of motivating and challenging tasks that learners will accomplish through social dialogue (Wheatley, 1991).

The Wheatley model is a problem-centered teaching model that is made up of three components: task, cooperative learning, and sharing. The Wheatley model is shown in Figure 1.

Figure 1

Wheatley Model Adapted (Wheatley, 1989).



The model begins with a task to be performed by students. The task is a concept or a topic that may cause a problem for students in their learning process. The task should consist of a set of problematic issues that address the core concepts of the subject. Such tasks facilitate the development of in-depth cognitive models (Dolmans & Schmidt, 2000), leading to the development of effective problem-solving skills, flexible knowledge, and self-directed learning skills (Hmelo-Silver, 2004). The provision of tasks to students provides an avenue for them to draw on their experiences to come up with uniquely personal solutions. Wheatley (1991) accentuated that "the core of problem-centered learning is a set of problematic tasks that focus attention on the key concepts of the discipline that will guide students to construct effective ways of thinking about that subject" (p.16). An appropriate and effective task should incorporate interdependence among students without compromising individual student accountability (Jolliffe, 2007). To successfully achieve the benefits of problem-centered learning, Wheatley (1991) argued that the task should motivate students to dialogue and communicate, make informed decisions, ask higher-order questions, be enjoyable, and encourage the transfer of learning.

Students work on the task in cooperative groups. Slavin (2011) explained cooperative group/learning as "an instructional method in which instructors organize students into small groups, which then work together to help one another learn academic material" (p. 344). Cooperative learning helps students to meet colleagues who have marginally advanced cognitive levels, one within the student's zone of proximal development (Applefield et al., 2000). The improvement of learning outcomes collectively by the group of learners is underpinned by the principles of positive interdependence where the attainment of individual outcomes is hinged on the success of other group members and individual accountability where every student contributes effectively to the group work (Abramczyk & Jurkowski, 2020; Johnson & Johnson, 1985; Jolliffe, 2007; Slavin, 1995).

The creation of positive interdependence and the contribution of individual efforts through individual accountability creates a social constructivist classroom (Applefield et al., 2000). Teachers believe that such interactions in cooperative learning foster the development of personal and social learning (Abramczyk & Jurkowski, 2020). Therefore, to achieve the utmost benefit of cooperative learning, students should not just be put in groups with the hope that they will work effectively

(Jolliffe, 2007; Veldman et al., 2020), but rather conscious efforts should be made to ensure that the task demands interdependence and accountability on individual students (Jolliffe, 2007).

Finally, the sharing stage seeks to give time to students to present their solutions, inventions, and insights (Wheatley, 1989). Students in various groups share their ideas with the class, whereby they learn to socially negotiate to come to an acceptable compromise when they disagree on answers and methods (Wheatley, 1991). Students' level of mastery and conceptualization of concepts improve when they can successfully explain concepts, methods, and answers to their contemporaries (Brooks & Brooks, 1999).

Moreover, students develop communication skills and master creative thinking as a result of effectively sharing their views with their peers. Here, the group shares their outcome or solution to the problem, and those listening develop the skill of honest talk and active listening respectively (Jolliffe, 2007). Ultimately, the sharing stage will facilitate the development of students' language skills and bring to the fore contentious issues and different perspectives on the solution to the task undertaken. (Kagan & Kagan, 2009). Students have lots of ideas to share, and when they master the completion of tasks in a group, the internalization of information arises for each person to diverse degrees, according to their personal experience.

Although the problem-centered approach is a student-centered learning strategy, teachers have their roles to play. The teacher observes and interacts with the students as they search for their information in their cooperative groups and provides support to the whole group instead of individual students (Wheatley, 1991). Teachers should serve as role models in terms of behavior expected to be seen in students during the enactment of problem-centered approaches (Veldman et al., 2020). Such behavior modeling should be seen throughout the learning period through the demonstration of facilitating skills and an impartial, non-judgmental role during the sharing period (Wheatley, 1991).

Since the Wheatley model is found to maximize students' learning in a collaborative environment, it will be prudent if its effectiveness is ascertained in the teaching and learning of genetics concepts in the Ghanaian educational context. In this regard, this paper reports the outcome of research conducted to explore the effectiveness of the Wheatley model in the teaching and learning of genetics concepts. The following hypotheses and research questions guided the research:

Ho_1: There is no statistically significant difference between the achievement scores of students exposed to the Wheatley model and students exposed to conventional instruction.

H_{02}: There is no statistically significant difference between the post-test scores of low achievers and high achievers when instructed through the Wheatley model.

Research question:

What are students' attitudes toward the Wheatley model as an instructional strategy?

Methods

Research Design and Procedure

The research design that was employed for this study was the quasi-experimental control and experimental non-equivalent group design since the subjects were not assigned randomly to the control and the experimental groups (Shadish et al., 2002; White & Sabarwal, 2014). The design was appropriate because it reduced the interactive effect of treatment and increased the external validity of the findings (Creswell & Plano Clark, 2011). In addition, the choice of quasi-experimental design for the study allows the investigation of intact groups in real-life classroom settings since it was not necessary to randomly assemble students for any intervention during the school hour to create artificial conditions.

Two senior high schools were randomly selected from the senior high schools that offer elective science subjects through the use of a table of random numbers. The school that was selected first was the experimental group, and the second school selected was the control group. In this study, the performance of students was the dependent variable, whereas the teaching strategies (the Wheatley model and a conventional approach) were the independent variables. The study used two separate treatments. The control group was taught through the conventional approach of teaching. In this approach, the teacher-led class interaction mostly explained concepts to students. The approach in this group was a typical lecture technique of teaching interspersed with questions to which students had to respond. The experimental group was taught the same topic using the Wheatley model. Both groups were taught simultaneously.

To ensure that there was no interaction effect, the selected schools were in different towns but within the same municipality. Since the teacher factor is an important variable of instruction, the same teacher taught both groups. It was ensured that the content to be delivered was the same, with the only differing attribute being the pedagogical approach through which the concepts were delivered. A post-test was conducted to ascertain the performance of both groups after the instruction. All the students in the experimental group were interviewed. Interviewees were given assurance of confidentiality and anonymity before the interview session. Express permission was also sought from the headmasters of the schools in which the study was conducted. The third layer of permission and consent was sought from the students in both groups. Students were assured of the confidentiality and anonymity of their responses, especially those who were interviewed. The interviewees were made aware that their responses would be recorded with an audio tape recorder.

Sample

Seventy-five senior high school year two students drawn from two intact classes from two randomly selected schools in the Bolgatanga Municipality of the Upper East Region of Ghana constituted the sample for the study. There were 41 students in the control group and 34 students in the experimental group. In each school, the simple random sampling technique was used to select one intact class for the study. The two selected classes from the two schools were categorized as experimental and control groups based on a pretest conducted after the classes were selected.

Instrument

Biology achievement tests (pre-test and post-test) and interview schedules were employed as the instruments for this study. The pre-test was based on first-year topics in the biology syllabus, which the students in both groups had been taught. The pre-test was used to ascertain whether the two groups were performing at the same level before the experiment and therefore comparable in terms of achievement. Thus, the pre-test was used to identify the entry characteristics of the students to determine if they shared similar traits and attributes. Again, since the focus of the paper was on students' performance on the post-test, it was deemed not appropriate to assess the students on the yet-to-be-treated concepts since the use of the same test items for pre-test and post-test could confound students' actual performance.

The pre-test was also used to group the students in the experimental group into low achievers and high achievers. Students with scores lower than the group mean on the pre-test were categorized as low achievers, and those with scores above the mean were termed as high achievers. Post-test was based on the topic of genetics, which was taught during the intervention to find out students' performance after the intervention. There were 25-item multiple-choice questions in the pre-test and the post-test. The pre-test and post-test items were used to find out the performance of the students before and after the intervention, respectively. The Kuder-Richardson (KR) 20 coefficient of reliability test was established for the achievement test items. The result indicated a reliability coefficient value of 0.7, indicating reliable test items for a classroom test. The KR 20 was used because the test items were multiple-choice questions and were scored either correct or incorrect. A semi-structured interview, which forms part of the instrument for the study, was used to find out how the students found the Wheatley model in terms of interest, understanding of the course content, and difficulties they might have encountered while working with the teaching model.

Data Analysis

Quantitative and qualitative analyses were used in this study. The independent sample *t*-test was used to analyze the quantitative data from the pre-test and post-test. The independent sample *t*-test was used to find out if there existed any statistically significant difference between the post-test scores obtained by students exposed to the Wheatley model as compared to the scores obtained by students exposed to conventional instruction. The independent sample *t*-test was again used to find out if there were any statistically significant differences between high achievers and low achievers on the post-test scores when they were taught using the Wheatley model. The responses from the interviewees during the interview session were themed, transcribed, and analyzed to unearth the attitude of the students toward the Wheatley model.

Results

The first null hypothesis sought to indicate that no significant difference would be found between the means of the post-test scores obtained by students exposed to the Wheatley model as compared to the scores obtained by students exposed to conventional instruction. To test this hypothesis, the independent sample *t*-test showed that the pre-test scores of the experimental group and the control group were not statistically significant ($t = 1.0441$, $df = 73$, $p = .299$) as can be seen in Table 1.

Table 1

Independent Sample t-test Analysis of the Scores of the Experimental and Control Groups on the Pre-test and the Post-test

	Group	***N***	**Mean**	**SD**	***df***	***t*-value**	***p*-value**
Group scores on the pretest	Experimental	34	12.26	2.400	73	1.0441	.299
	Control	41	12.90	2.827			
Group scores on the post-test	Experimental	34	15.97	1.714	73	4.694	.000
	Control	41	13.17	3.106			

Significance *p<0.05

This indicates that there was no difference in performance between the two groups before the study was conducted. The *t*-test results, however, showed a statistically significant difference between the two groups ($t = 4.694$, $df = 73$, $p < .001$) in post-test scores of the achievement test. This information is displayed in Table 2. The null hypothesis is therefore rejected, indicating that the Wheatley model (experimental) group performed better with a Mean of 15.97 and SD of 1.714 than those taught by the conventional approach (control) group with a Mean of 13.17 and SD 3.106. The outcome indicates a large boost for the use of the Wheatley model in science classrooms.

The second hypothesis, that there is no statistically significant difference between the post-test scores of low achievers and high achievers when instructed through the Wheatley model, was tested with the independent sample *t*-test and the results are presented in Table 2.

Table 2

Results of the Independent Sample t-test Analysis on the Pretest Scores of High Achievers and Low Achievers in the Experimental Group

Students score on	Achievement level	*N*	Mean	SD	*t*	*df*	*p*-value
Pretest	High achievers	23	12.26	1.287	-6.853	32	.000
	Low achievers	11	9.18	1.079			
Posttest	High achievers	23	16.35	1.774	-2.308	32	.028
	Low achievers	11	15.00	1.095			

Significance *p<0.05

At the onset of the research, there was a significant difference between students categorized as high achievers and those categorized as low achievers in favor of the high achievers, as can be seen in Table 2. There is an indication that high achievers were performing better than low achievers before the study was carried out, with a higher mean score. On the post-test, the results from the independent sample *t*-test were still statistically significant (t = -2. 308, p = .028) on the post-test as can be seen in Table 2. This implies that when students are taught using the Wheatley model, high achievers will continue to perform better than low achievers. The null hypothesis of no significant difference between the performance of high achievers and low achievers, when taught using the Wheatley model, is therefore rejected.

The research question sought to identify students' attitudes toward the teaching and learning strategy. Students' voices are very important in the introduction of any new strategy. Thus, the students who were instructed with the Wheatley model were interviewed to gauge their attitude toward the instructional approach. Since attitudes are a multidimensional construct with different sub-constructs (Kind et al., 2007; Osborne et al., 2003), students' responses to the interviews were grouped into themes reflecting different aspects of their attitudes towards the teaching strategy. The themes that emerged from the interviews were excitement about the learning process, understanding of concepts, collaboration in the teaching process, and instructional time.

Osborne et al. (2003) identified the 'enjoyment of the learning process' as an aspect of students' attitudes toward science. Learners draw on their prior experiences to learn new concepts therefore, if the learning situation is not exciting to them, the likelihood of engaging in further learning is minimized. The teaching and learning process should be enjoyable to the student, especially when a new teaching approach is being introduced. The majority of the students who were instructed with the Wheatley approach found it to be interesting and exciting. The students found the opportunity to explore and contribute to the learning process very exciting. Students' views were typified by comments such as those presented as follows:

> Student 1" I was very excited because the approach has helped some of us to contribute well in class and that was interesting".
> Student 4 noted that the approach was very good "because you can contribute and share ideas in class".
> Student 7 "I was very happy because I had to search for information before presentation which was exciting because we don't do that in class".

Teachers teach with the hope that their students will understand science concepts, which will lead to increased and improved achievement in the scores of students. 'Achievement in science' has therefore been identified as an aspect of the overall attitudes of students toward science (Osborne et al., 2003). The difference in the mean scores of the students in the Wheatley instructional group and students in the control group was statistically significant; nonetheless, it was necessary to gauge students' views on their conceptual understanding when instructed with the Wheatley model. Students expressed how the approach enabled them to understand the concepts taught. "I understood the concepts very well," Student 2. She continued that "learning with peers is much (sic) interesting and helps in understanding than with the teacher". Student 3 sought to explain the reason why he understood the concepts by indicating that "we interacted and argued to come out with correct explanations".

A cardinal attribute of any constructivist approach is social interactions among students (Liang & Gabel, 2005). Since the Wheatley model is constructivist, the social interaction component needs to be realized. Students were asked to indicate their views on the interactions they had during the teaching and learning process. Students indicated that they were able to share ideas with their peers as they were exposed to the Wheatley model. Student 3 voiced that he was able to share ideas, "especially where terms were not clear to the group and wouldn't have understood everything alone". Student 6 stated that "I shared ideas with colleagues, some words are different in spelling but the same in meaning" and that she wouldn't have wished to learn genetics alone "because I would have had information in one direction". Student 10 stated that "group learning is better than sitting in class to learn with a teacher".

The final aspect of student attitudes toward the approach used in this study was instructional time. It was prudent that students' views towards the duration of the learning process were gauged to alert teachers. Some of the students indicated that the time allocated to them to search for information related to the concepts and report back to the class was not enough. Student 1 said, "I wish we had more time." Student 4 noted that "time was not enough". Student 5, on his part, answered that the "time was not that adequate". Instructional time is a critical aspect of the variables that influence the outcome of the teaching and learning process. When students and teachers do not have enough time to facilitate the teaching and learning process, the most salient concepts are ignored or rushed through. The consequence of not either completing the content material or covering it superficially is that students' conceptual understanding is likely to be impaired.

Discussion

The outcome of the study indicates that students taught through the Wheatley model performed better than those taught through the conventional approach. This result is similar to that obtained by Wheatley (1989), who found the approach superior to the conventional teaching strategy in terms of students' academic achievement. Wheatley (1991) argued that students' higher academic achievement could be because when teachers set activities for students, it forces the restructuring of ideas at a higher level than using the explain-practice paradigm. Kim (2005) found that the use of a constructivist approach improved students' achievement when compared to the conventional teaching strategy. The various forms of the constructivist approaches have been found to generally have superior ability than the conventional approach in improving students' conceptual understanding and overall learning outcomes (Liang & Gabel, 2005).

Students' responses during the interview session indicate a very high level of satisfaction and positive attitude towards the use of the Wheatley model. There was increased student participation, which brought out diverse views on the learning process. Wheatley (1989) noted that "knowledge is not passively received but is actively built up by the cognizing subject and to know is to understand in a manner which can be shared by others (p. 164)". The increased student participation and excitement can be attributed to the fact that students could challenge the views of their colleagues through collaborative efforts. This provided a conducive environment for them to learn, which was in line with suggestions made by Wheatley et al. (1995), indicating that students learn best when they are provided with the appropriate challenging problem in a collaborative environment. Liang and Gabel (2005) noted that when constructivist approaches are used in class, there is active participation by students during learning, and this elicits students' interest in the content even when it seems difficult to them, and they will express themselves freely as they work in groups.

The responses from the interviews revealed that the students not only enjoyed the teaching and learning process, but they also understood the concepts. Although the achievement test proved this assertion to be true, it was revealed that students articulated that they understood the concepts. Collaborative group learning seems to improve students' learning because opportunities are provided for students to contribute and learn through their colleagues' views, which increases motivation (Yu-Chien, 2008). Since learning is maximized in social interaction through individual construction (Bauersfeld, 1988), the students instructed with the Wheatley model had the opportunity to communicate and exchange ideas with their colleagues, which proved to be very positive for their learning. Social interaction within the classroom has been found to have a positive impact on students' achievements, attitudes, and motivation to learn (Liang & Gabel, 2005). Although Liang and Gabel (2005) asserted that some students fail to express themselves and confront cognitive conflicts in constructivist classrooms, this research did not encounter such problems. The students in this study were able to express themselves by either disagreeing or agreeing with their colleagues without any difficulty.

The downside of the approach was the seeming lack of time for students. Although the approach was to allow students to work at their own pace, they had to do so within the stipulated time allotted for the subject on the school's timetable. Airasian and Walsh (1997) noted that students will need different durations to construct meanings due to their different ability levels. Teachers are expected to provide adequate timeframes to cater to the uniqueness of each student. Unfortunately, durations for instructions are delineated by the school system, and therefore, each teacher has to use the time allocated to them. This makes the use of constructivist strategies laborious in certain circumstances. In this study, some students expected to use more time when they were instructed with the Wheatley model.

Conclusion

It can be concluded from the findings of this study that the use of the Wheatley model as a teaching and learning intervention is found to be more effective than the conventional approach of teaching in terms of student achievement. The Wheatley model, however, was not able to improve the performance of low achievers within the classroom. It can also be concluded that students' attitude towards the Wheatley model as a teaching and learning strategy was positive and that the approach improved collaborative learning among students learning with the Wheatley model.

Implication for Science Education

The outcome of this research provides yet another constructivist approach that science teachers can use to enact scientific concepts to their students. The research has provided evidence that students taught using the Wheatley model are more likely to perform better than those using the conventional approach. Thus, since teachers are seeking various pedagogical ways through which to represent content knowledge to students to maximize learning with its associated improved academic performance, the Wheatley model can be used in science classrooms.

The fact that students showed positive reactions to the approach provides great promise in the use of the Wheatley model in science classrooms. Such excitement shown towards the approach can ultimately affect the general attitude towards the concepts in particular and the subject in general. Thus, in an era of decreasing and declining interest in science, the use of the Wheatley model can provide an avenue through which students' interest in science can be whipped up. The emphasis on collaborative group work in the approach, which the students enjoyed and noted its influence on their learning, is a critical attribute that is very much needed in this generation. Thus, teachers can foster cooperation and emphasize collaborative attitudes and tendencies among their students using the Wheatley model.

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Towards Improved Geometry Instruction: Learners' Experiences with Technology-Enhanced and Conventional Van Hiele Phased Instruction

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ABSTRACT

This study investigated learners' experiences and understanding of transformation geometry using two instructional strategies: Conventional Van Hiele Phased Instruction (CVHPI) and Technology-Enhanced Van Hiele Phased Instruction (TVHPI), incorporating GeoGebra as a digital tool. Through semi-structured interviews, qualitative data were collected from 48 Senior Three secondary school learners who participated. Thematic analysis revealed that TVHPI, supported by GeoGebra, enhanced visual learning and dynamic interaction with geometric concepts, though learners faced technical challenges and limited practice time. CVHPI, while providing structured and step-by-step instruction, particularly benefited lower achievers but was less effective in addressing complex misunderstandings. As a result of this study, a Geometry Pedagogical Improvement Cycle (GeoPIC) framework was developed to improve the teaching and learning of geometry through a continuous and systematic process. The GeoPIC framework emphasizes adopting instructional strategies, tailoring them to individual needs, aligning with learner expectations, and incorporating feedback through a cyclical reflection and adjustment process. This study highlights the potential of combining technology-enhanced tools with conventional instruction and presents GeoPIC as a model for refining pedagogical approaches in geometry education.

Keywords: Transformation Geometry, learners' experiences, Van Hiele levels, Technology, GeoGebra, and GeoPIC.

Introduction

It has long been acknowledged that geometry instruction is an essential part of mathematics since it provides fundamental knowledge for various fields, such as computer science, engineering, and architecture (Çavuş & Deniz, 2022; Smith & Jones, 2020; Sunzuma & Maharaj, 2019). Transformation geometry is one of the numerous subfields of geometry that is particularly essential because of its real-world applications and ability to help develop spatial reasoning and problem-solving skills. However, despite its importance, transformation geometry can be complex for learners to understand using traditional teaching techniques due to the subject's abstract nature (Bradley, 2005; Brijlall & Abakah, 2022; Ndungo et al., 2024).

In this regard, several studies have brought attention to the ongoing challenges that learners encounter when trying to grasp geometric concepts; these challenges frequently result in worry, anxiety, and eventually low academic achievement among learners. Consistently low achievement levels in geometry have been reported in countries including Indonesia, South Africa, Nigeria, Pakistan, Sri Lanka, and Italy (Ayebale et al., 2020; Ubi et al., 2018; Silmi Juman et al., 2022). These results highlight a widespread issue and critical need for educational interventions to raise geometry competency among learners (Ayebale et al., 2020).

Ngirishi and Bansilal (2019) have observed that many learners cannot comprehend basic geometry concepts, analyze geometric properties, and recognize shapes. As a result, children frequently function at lower geometric thinking levels, which hinders their capacity to understand more complex ideas and progress to advanced levels of geometric thinking. Additionally, educators frequently use conventional teaching techniques, which might not be able to meet all of their students' learning demands (Kivkovich & Chis, 2016). This antiquated method may cause learners to disengage and impede their comprehension of fundamental geometric concepts.

Erroneous beliefs and unfavorable perceptions regarding geometry intensify these difficulties. Many learners find it challenging to apply formulas and theorems, evaluate arguments, and comprehend geometric vocabulary, making geometry seem challenging and uninteresting (Kivkovich & Chis, 2016). Furthermore, learners frequently struggle with mathematical tools like compasses and protractors, leading to missed questions and further obstacles with problem-solving (Luneta, 2015). To tackle these problems, Moru et al. (2021) suggested that a concentrated effort must be made to improve conceptual understanding and implement effective instructional strategies. Enhancing geometry instruction can help learners better understand the topic and improve their general mathematical aptitude.

Educational theorists have investigated various teaching strategies to improve learners' geometric thinking and overcome these issues. In particular, the Van Hiele Phased Instruction model is well known for its precise approach to teaching geometry. This model guides learners from fundamental shape recognition to more complex reasoning about geometric features and transformations by emphasizing the progression through five levels of geometric understanding. Even though Conventional Van Hiele Phased Instruction (CVHPI) has proven to be successful in assisting learners in understanding, there is increasing interest in incorporating technology into this framework to improve learning results even more (Machisi & Feza, 2021; Zalman, 1982).

Moreover, Technology-Enhanced Van Hiele Phased Instruction (TVHPI) using programs like GeoGebra allows learners to interact actively and visually with geometric concepts. With GeoGebra's interactive visualization features, learners can modify geometric shapes, see transformations in real time, and more concretely investigate the links between geometric objects. This method has the potential to close the knowledge gap between learners and abstract geometric concepts, increasing the accessibility and interest level of transformation geometry (Adelabu et al., 2022; Iannone & Miller, 2019; Mthethwa et al., 2020; Vágová & Kmetová, 2019; Ndungo, 2024).

Technology integration in education has expanded significantly, aligning with curriculum trends emphasizing active learning through interactive tools like tablets, smartphones, and specialized software (Diaz-Nunja et al., 2018). Learners with information and communication technology tools support their integration into mathematics education, providing more significant learning opportunities and fostering engagement and discovery-based learning (Mosese & Ogbonnaya, 2021). Technology integration in teaching geometry is further emphasized by Uganda's new lower secondary curriculum and other recent related studies (National Curriculum Development Center, 2019; Ndungo et al., 2025).

Research highlights the benefits of GeoGebra in enhancing geometric reasoning and engagement. For example, Abdullah and Zakaria (2013) found that learners using dynamic software like GeoGebra made significant progress in geometric understanding compared to traditional methods. Other studies indicate that GeoGebra supports visualization, spatial reasoning, and problem-solving, making mathematics more engaging and enjoyable (Mollakuqe et al., 2020; Celen, 2020). It has also proven effective in teaching diverse topics, including circles, linear functions, 3D geometry, and trigonometry, helping learners explore mathematical concepts more thoroughly (Mudaly & Fletcher, 2019; Uwurukundo et al., 2021; Yildiz & Baltaci, 2016). Over the past few decades, there have been substantial changes in how geometry is taught. Increasingly, the emphasis is on developing conceptual understanding rather than rote memorization of processes (Clements & Sarama, 2021). Formal deduction has supported traditional geometry teaching methods, yet it is less successful for younger learners or those without prior knowledge of geometric ideas (Van de Walle et al., 2016). Early in life, developing spatial thinking and visualization abilities is crucial to comprehend more complex geometric ideas like transformation geometry (Sinclair & Bruce, 2015).

However, despite these advancements, the literature reveals ongoing challenges in effectively teaching transformation geometry, particularly in contexts where learners struggle with abstract concepts like rotations, reflections, and dilations (Sunzuma & Maharaj, 2019). Silmi Juman et al. (2022) noted that applying geometric theorems and resolving complex problems are two significant obstacles that learners encounter when learning geometry. Their research demonstrated how activity-based learning strategies might help learners overcome these challenges and increase their comprehension and engagement.

Similarly, the current study furthers this foundation by examining the contribution of teaching strategies such as CVHPI and TVHPI to improving learners' understanding of transformation geometry. This study investigates the additional effects of incorporating technology, notably GeoGebra, into the learning process, whereas Silmi Juman et al. (2022) concentrated on active learning methodologies. This study closes a significant knowledge gap by investigating the use of technology-enhanced learning in the Ugandan environment. It offers insights into how such tools can address enduring difficulties in geometry teaching.

Theoretical Underpinning

This study is grounded in Cognitive Constructivism (Piaget), emphasizing active learning through progression in the Van Hiele Model of Geometric Thought. It integrates Social Constructivism (Vygotsky), highlighting the role of social interaction and teacher scaffolding within the Zone of Proximal Development (Allen, 2022). This study also incorporates Technology-Enhanced Learning, using GeoGebra to support interactive and visual learning in transformation geometry. The Van Hiele theory provides a framework for understanding the progression of geometric thinking through five levels: Visualization, Analysis, Abstraction, Deduction, and Rigor (Crowley, 1987; Vojkuvkova, 2012). The theory further proposes a five-phased instructional approach to geometry that aligns with these cognitive stages, ensuring that teaching methods are suited to the learners' current level of understanding (Abdullah & Zakaria, 2013; Moru et al., 2021).

Instructional Strategies

This paper explores how the two instructional strategies, CVHPI and TVHPI, influence learners' experiences and understanding of transformation geometry. The following sections detail these instructional strategies.

The Van Hiele's Phased Instructional Strategy (CVHPI)

The Van Hiele Phased Instruction (Conventional Van Hiele Phase Instruction in this study) is a structured framework for teaching geometry to support learners' understanding and progression through the Van Hiele levels. This model guides educators in designing lessons that help learners advance through increasingly complex levels of understanding with teacher support (Bonyah & Larbi, 2021; Machisi & Feza, 2021; Moru et al., 2021; Pujawan et al., 2020; Tahani, 2016). Figure 1 shows the five phases of instruction according to the Van Hiele Phased Instruction model.

Figure 1

Showing the Five Van Hiele Phases of Instruction

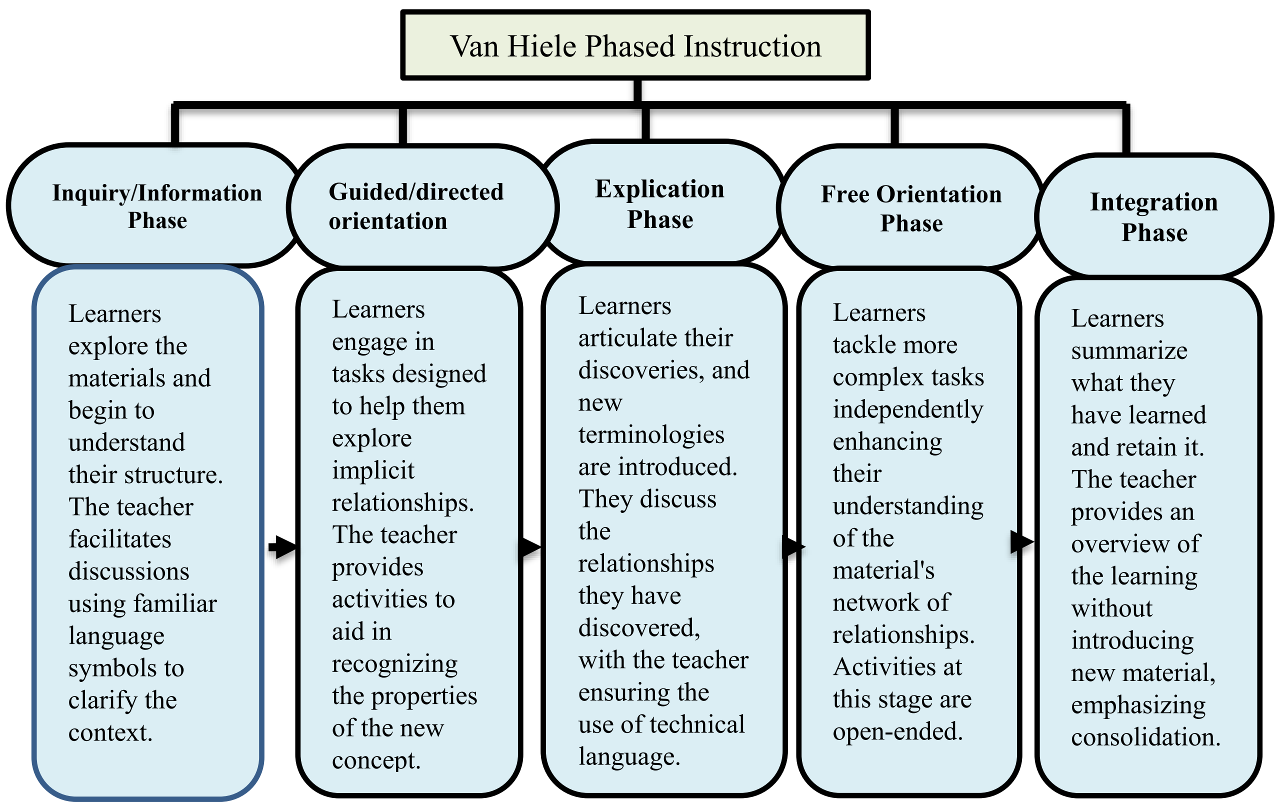


Figure 1 illustrates the five phases of the CVHPI: (1) Inquiry, where learners explore concepts; (2) Directed Orientation, focusing on exploration activities; (3) Explication, where learners articulate understanding; (4) Free Orientation, applying knowledge to complex problems; and (5) Integration, consolidating concepts for deeper comprehension.

Technology-enhanced Van Hiele Phased Instructional Strategy (TVHPI)

Building on the principles of CVHPI, TVHPI integrates technology into the learning process. The current study used GeoGebra to enhance visualization and interactivity in geometry lessons. TVHPI aims to engage learners more effectively by offering dynamic representations of geometric transformations, fostering an interactive and collaborative learning environment. By incorporating GeoGebra, TVHPI addresses diverse learning styles and aims to improve learner motivation and engagement in geometry education (Uwurukundo et al., 2021). Figure 2 shows a GeoGebra environment.

Figure 2

GeoGebra Environment: Showing How to Find the Angle of Rotation Given Two Rotated Shapes.

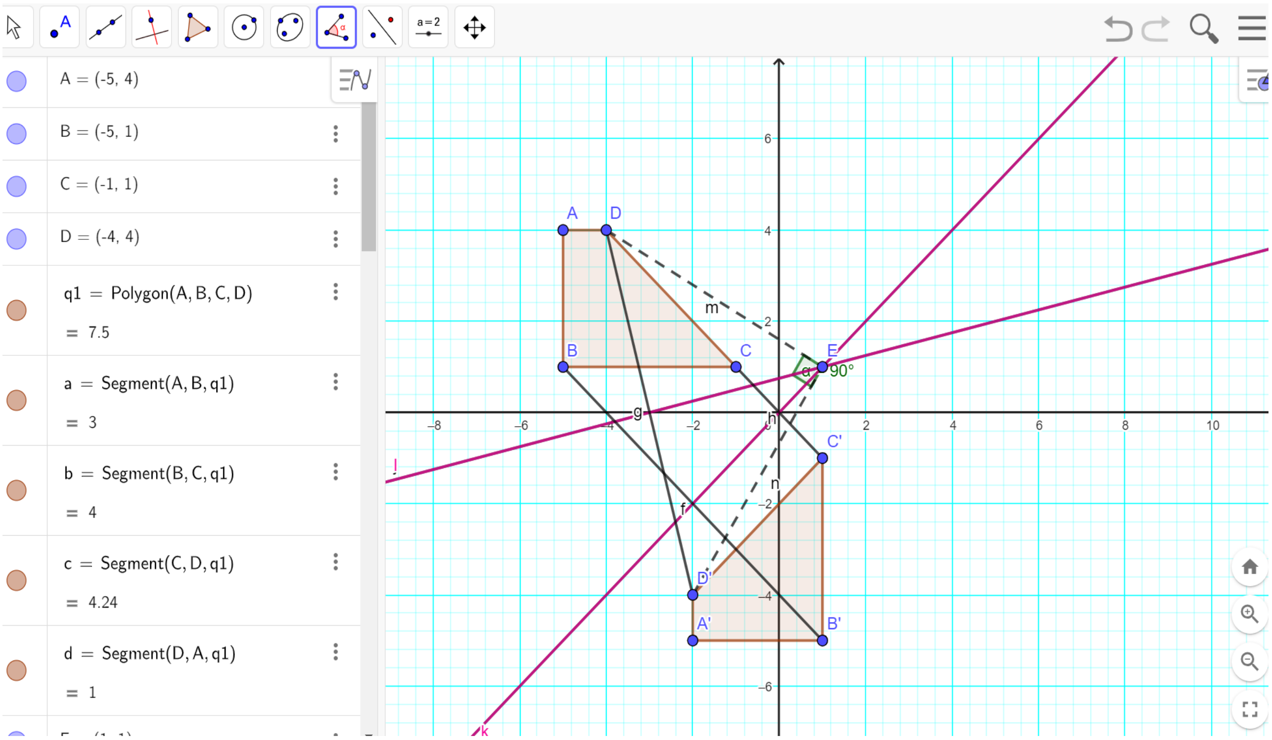


Figure 2 depicts the GeoGebra interface with a GeoGebra-generated diagram for finding the rotation angle given two rotated shapes. The figure demonstrates line segments, perpendicular bisectors, and an angle of rotation of 90^0. GeoGebra tools allow users to dynamically adjust rotation angles, reflection lines, enlargement scale factors, and observe real-time changes, enhancing their understanding of transformations. The interface has two interactive environments, the analytical and the graphical, that support the analytical and graphical analysis of geometric shapes and transformations. This enables learners to explore geometric transformations with precision and clarity.

The key differences in how teachers can implement CVHPI and TVHPI lie in the mode of preparation, instruction, learner activities, visualization, feedback, and assessment in transformation geometry. For example, CVHPI relies on physical tools such as graph paper, mirrors, and rulers, requiring manual demonstrations and learner activities focused on drawing, plotting points, and performing calculations that lead to transformation. In contrast, TVHPI utilizes GeoGebra, enabling dynamic, real-time visualization of transformations. While CVHPI provides static visualization through blackboard drawings and physical manipulatives, TVHPI encourages interactive exploration. Feedback in CVHPI is given manually, with teachers reviewing learners' work, whereas TVHPI delivers instant feedback via GeoGebra's outputs. CVHPI operates at a teacher-controlled pace with repetitive exercises, while TVHPI supports self-paced learning through interactive tasks. Exploration in CVHPI is guided by limited opportunities for independent discovery, unlike TVHPI, which promotes learner experimentation and exploration of concepts. In CVHPI, the teacher's role is directive, leading each step, while in TVHPI, the teacher acts as a facilitator. In CVHPI, assessment focuses on procedural accuracy in written and drawn work, whereas TVHPI evaluates learners' ability to perform and interpret transformations digitally.

Problem Statement

The Van Hiele Phased Instruction model has been extensively explored and proven to be a practical approach to teaching geometry, particularly in fostering learners' geometric thinking (Machisi & Feza, 2021; Narh-kert & Sabtiwu, 2022; Office of the Prime Minister, 2020; Savec, 2019). However, learner engagement and conceptual understanding challenges persist despite its success, especially with complex transformations. In the Ugandan context, limited research has been conducted on the implementation and effectiveness of CVHPI, leaving a gap in understanding how this model functions within local educational environments. With the current trend in education emphasizing technology integration to enhance learning outcomes, there is a growing need to investigate how CVHPI can be improved by incorporating technology-enhanced tools like GeoGebra. While GeoGebra has shown potential to support visualization and engagement, its impact when combined with CVHPI has not been thoroughly examined in Uganda. This study seeks to address this gap by exploring how TVHPI can further support learners' learning, improve understanding, and help overcome the difficulties learners face in geometry.

Research Questions

The main research question is: How do learners' experiences and understanding of geometry differ between TVHPI and CVHPI?

The research questions are: (1) what are the challenges and support needs associated with TVHPI and CVHPI? Furthermore, (2) what is the learners’ perception of the effectiveness of CVHPI and TVHPI in enhancing their understanding of transformation geometry?

Methodology

Research Design

The research followed a quasi-experimental design, but this paper's primary focus is on the qualitative component of the main study. Semi-structured interviews were used to gather detailed data on learners' experiences with CVHPI and TVHPI. The qualitative design allowed an in-depth exploration of how each instructional strategy impacted the learning of transformation geometry. The study provided a robust framework for examining the differences in learners' experiences and engagement with transformation concepts by comparing two distinct instructional approaches within the same learning environment.

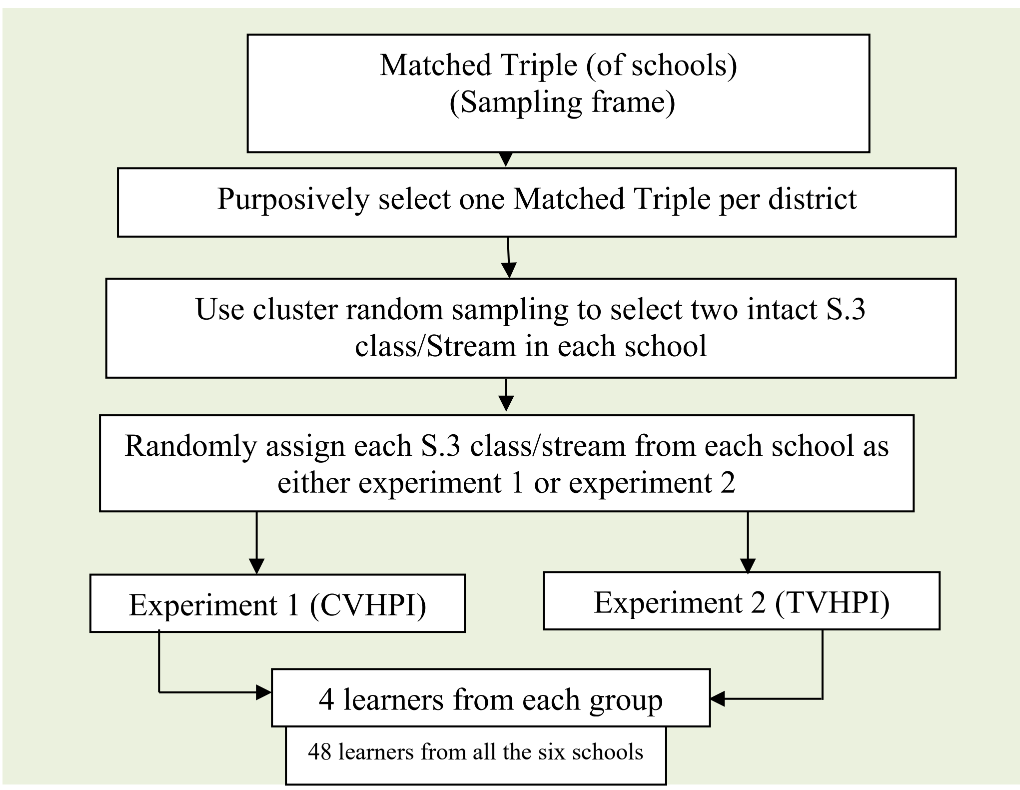
Study Population, Sampling, and Sample

The population for this qualitative component comprised Senior Three (S.3) learners attending secondary schools in both a rural and an urban district within Midwestern Uganda. The two districts (one rural and one urban) were selected for their contrasting settings and diversity in educational contexts. According to 2024 data from the respective education departments, the rural district had 98 secondary schools with an average of 80 S.3 learners per school, yielding an estimated 7,680 learners. The urban district had 19 secondary schools with an average of 150 S.3 learners per school, resulting in approximately 2,850 students. This gave a combined population of 10,530 S.3 learners. The selection of S.3 learners was purposive due to their advanced engagement with the geometry curriculum. By this level, learners have built foundational knowledge in geometric concepts and reasoning, making them suitable participants for examining the impact of instructional strategies on their understanding of transformation geometry. Their maturity and cognitive readiness further contributed to the study's feasibility and potential for insightful outcomes.

The study employed a combination of matched and purposive sampling techniques. Initially, schools within the two districts were assessed against inclusion criteria, such as administrative flexibility, the availability of multiple S.3 streams, and access to computer laboratories. Matched triples were created to ensure comparability across selected schools, with one triple (three schools) chosen from each district. Two senior classes in each school were randomly assigned to experiment one (CVPHI) or two (TVHPI) groups. Thus, CVHPI and TVHPI were implemented in the same schools to control external variables. The critical difference was the instructional method: Group content sharing had no significant effect as the hands-on, technology-enhanced experience remained exclusive to TVHPI. Figure 3 shows the sampling procedures that were followed in the study.

Figure 3

Showing the Flow of the Sampling Procedures



Initially, 651 learners from six schools participated in the broader quasi-experimental study (317 from rural schools and 334 from urban schools). Following attrition due to dropouts and incomplete data, 483 learners (245 in CVHPI, 238 in TVHPI) were retained for the final analysis of the learners' Van Hiele levels (quantitative results on Van Hiele levels and attitudes are not within the scope of the current article). For the qualitative phase, a purposive subsample of 48 learners was drawn from the larger sample, selected based on their Van Hiele Levels. Ensuring the inclusion of diverse learners equally distributed across gender (24 males and 24 females), location (24 urban and 24 rural), achievement level (24 lower achievers and 24 higher achievers), and instructional strategies (24 taught using TVHPI and 24 using CVHPI).

Data Collection Instruments

The primary data collection tool for the qualitative component was semi-structured interviews that allowed for flexibility in responses while maintaining a consistent structure, ensuring that key themes were addressed across all participants. The researchers designed the interview questions to explore four main areas: Learning Experience, Instructional Impact, Teaching Effectiveness, and Difficulties Encountered. This approach allowed learners to share their personal experiences in detail while ensuring that the researchers could gather comparable data across participants.

Procedures

The study began with a two-day training program to equip teachers with the necessary skills to deliver lessons using the TVHPI and CVHPI methods. The training covered the theoretical framework and practical application of the Van Hiele Phases of instruction, focusing on how to guide learners through different levels of geometric thinking. Teachers were also trained in using GeoGebra to teach transformation geometry, including how to construct geometric figures, apply transformations such as reflection, rotation, translation, and enlargement, and visualize geometric relationships dynamically.

To ensure consistency in lesson delivery, the first author and the teachers collaboratively developed standardized lesson plans covering all learning outcomes specified for the intervention. These plans detailed instructional content, learner activities, and lesson organization, ensuring that all participating teachers followed a uniform approach. Following the training, pretests were administered to the participating learners to assess the learners' baseline Van Hiele levels (achievement) and attitudes toward transformation geometry, so that subsequent changes could be attributed to the instructional methods used.

The intervention covered six weeks, with both groups (CVPHI and TVHPI) receiving four lessons of 40 minutes per week, amounting to 24 lessons by the end of the intervention, delivered following the regular school timetable. To maintain consistency, the same teacher instructed both groups while the first author monitored and supported teachers during the intervention to ensure fidelity. The experiment two group (TVHPI) received lessons from the computer laboratory, using GeoGebra to facilitate interactive and dynamic learning experiences. The setup for GeoGebra was installed on all the computers in the school computer laboratories; the teachers used projectors to deliver lessons, while the learners used the computers to perform different transformations. This group's first lesson was designed to introduce learners to the GeoGebra software. In contrast, the experiment one group (CVHPI) was taught in regular classrooms without technology; the teacher used a chalkboard set to illustrate transformations, while the learners used graph books/papers and mathematical set instruments to perform transformations.

At the end of the intervention, post-tests were administered to all learners to measure any changes in their attitudes and achievement following the six weeks of instruction. Finally, interviews were conducted one week after the intervention concluded. These interviews provided qualitative insights into learners' experiences and understanding of the geometry content, allowing the research team to gather in-depth information beyond the quantitative test results. The 30- to 45-minute interview sessions were held early in the mornings and the evenings to prevent interfering with regular lessons. The current paper presents only results from the qualitative part of the main study.

Data Analysis Methods

The data were analyzed using thematic analysis, following the six-step process outlined by Lapolla (2020) and Meyer and Avery (2009). First, the researchers familiarized themselves with the data by reading and re-reading the transcripts, allowing them to gain an immersive understanding of the content. Following this, anonymization of data was done using a system of unique identifiers based on the learner's gender, achievement level, school location, and method of instruction (e.g., MHA/TVHPI/R1 for a male(M), higher achiever(H) from a rural school(R1), taught using TVHPI). With great care, each transcript was divided into more manageable, insightful segments, with each unit illustrating a distinct subject, concept, or experience related to the study questions. The process yielded 438 different data units, allowing for a more concentrated and in-depth examination since each data unit represented a distinct component of the participants' responses, including their experiences with instructional strategies or the difficulties they had while learning geometry (Lapolla, 2020; Meyer & Avery, 2009). After the data was split and anonymized, it was organized, systematically coded, categorized, and thematically analyzed using Excel. Although our primary analysis is qualitative, utilizing pivot tables allowed for a comprehensive examination of the learners' experiences and the effectiveness of the instructional strategies by summarizing and highlighting links between themes, categories, and codes (Miller, 2014; Ngulube, 2015). This ensured a data-driven approach in drawing findings and offering recommendations by providing a concise summary of the data that served as a basis for examining the learners' experience of instructional strategies and transformation geometry.

Ethical Considerations

The study was conducted under ethical approval from the Mbarara University of Science and Technology (MUST) Research Ethics Committee and the Uganda National Council for Science and Technology (approval numbers MUST-2024-1519 and SS2857ES). Informed consent was obtained from parents or guardians, and assent was sought from learners themselves. For participants aged 18 and above, informed consent was obtained directly. All data were anonymized using unique identifiers to protect participants' privacy, and learners were informed of their right to withdraw from the study at any time.

Several strategies were employed to ensure the trustworthiness of the findings. Credibility was enhanced through member checking, where participants were invited to review their transcripts and confirm the accuracy of their responses. This process ensured that the findings accurately reflected the participants' experiences. Additionally, peer debriefing provided an external check on the data analysis process, as colleagues reviewed the coding and theme development to ensure the findings were consistent with the data.

Dependability was maintained through an audit trail, documenting all decisions made during the research process, from data collection to analysis. This audit trail provides a clear record of the steps taken, allowing the research process to be replicated in future studies. Finally, confirmability was established through reflexivity, where the researchers engaged in self-reflection throughout the study to identify and mitigate any potential biases and control the researchers' potential influence of professional backgrounds in mathematics education. This process helped identify potential biases and ensure they did not unduly influence the analysis. The researchers' belief in the potential benefits of technology-enhanced learning, for example, was explicitly acknowledged as a possible source of bias, and steps were taken to remain objective throughout the study. Lastly, CVHPI learners were granted one week of access to GeoGebra after the study concluded, ensuring ethical fairness in technological exposure.

Results

This study investigated the differences in learners' experiences and understanding of transformation geometry between two instructional strategies: CVHPI and TVHPI. In particular, it looked at how each approach affected learners' understanding of important geometric transformations, including rotations, reflections, and matrix transformations. In addition, the study sought to evaluate how well these strategies supported students' learning and dealt with issues that came up during teaching.

The sample consisted of 48 learners, equally distributed across gender, location, achievement level, and instructional strategies. Before analysis, the data were anonymized, with each learner assigned a symbol representing their contextual characteristics, such as gender, location, achievement level, and instructional strategy, rather than using names or identification numbers. This ensured confidentiality while maintaining relevant context for analysis. The data was organized in Excel and thematically analyzed. Two key themes emerged from the learners' experiences with CVHPI and TVHPI in learning transformation geometry.

Theme 1: Challenges and Support Needs

This theme explores learners' barriers to understanding geometry and the interventions required to address them. It emerges from three categories: *Learning Challenges and Obstacles, Metacognition and Self-Regulated Learning, and External Learning Support.* These categories highlight the interplay between procedural and conceptual difficulties, the role of self-regulation and reflection, and the importance of external assistance. Learning challenges often arise from difficulties in following steps, grasping abstract concepts, or accessing adequate resources. Metacognition involves learners' ability to focus, reflect on progress, and maintain motivation, which is influenced by instructional methods. External learning support emphasizes the need for guidance, tailored interventions, and structured practice to help learners navigate these challenges effectively. The theme highlights the complexity of overcoming learning barriers and the critical role of internal strategies and external resources.

Theme 2: Instructional Effectiveness and Learner Satisfaction

This theme captures how instructional methods impact learning outcomes and align with learners' needs. It evolves from five categories: *Instructional Support and Guidance, Learning and Differentiated Instruction, Practical Learning and Reinforcement through Practice, Role of Instructional Strategy in Learning, and Learners' Expectations*. These categories highlight the importance of effective guidance, tailored instruction, practical application, and structured strategies in facilitating understanding. Instructional methods aim to resolve misunderstandings, enhance visual learning, and provide opportunities for step-by-step reinforcement. Additionally, aligning instructional approaches with learners' expectations is essential for fostering engagement and satisfaction. This theme emphasizes the interconnectedness of instructional quality, strategy, and learner perceptions in shaping meaningful learning experiences in geometry. The detailed findings for each of these themes are discussed in the preceding sections.

The Challenges and Support Needs Associated With TVHPI and CVHPI

This section looks at the challenges and support requirements associated with CVHPI and TVHPI. It draws attention to learners' challenges in understanding geometry and the help needed to improve their learning through hands-on training and individualized guidance. Table 1 illustrates the occurrence of different codes in the data for each category in theme one.

Table 1

Showing a Count of Codes and Categories (Theme 1)

Row Labels	TVHPI	CVHPI	Total
Challenges and Support Needs	**77**	**89**	**166**
Learning Challenges and Obstacles	**49**	**60**	**109**
Challenges in Seeking Help	1		1
Difficulty Following Steps		3	3
Struggling with Geometry Concepts	16	38	54
Technical and Resource Challenges	32	19	51
Metacognition and Self-Regulated Learning	**2**	**9**	**11**
Concentration	2		2
Interest in Learning Geometry Concepts		2	2
Self-Reflection		7	7
External Learning Support	**26**	**20**	**46**
Need for Extensive Use of GeoGebra for Mastery	6		6
Need for Individualized Support and Time for Practice	20	20	40

Learning Challenges and Obstacles

Based on the data values in Table 2, learning challenges and obstacles were the most common experiences for learners in both instructional techniques, with 109 occurrences (49 in TVHPI and 60 in CVHPI). All learners encountered different challenges while trying to grasp transformation geometry. These challenges were classified as *struggling with geometry concepts, technical and resource challenges, and difficulties seeking help during the lesson.*

Struggling with geometry concepts was more pronounced in the CVHPI group, with 38 responses mentioning this difficulty compared to 16 in the TVHPI group. For example, one TVHPI participant noted: "Reflections were easier for me, but when we started working with enlargements, I got lost because I did not understand the scale factors" (FLA/TVHPI/R2). Similarly, a CVHPI participant shared: "But rotations were hard for me when we had to apply them on the grid. More examples would have helped" (FHA/VHPI/U2). The more significant number of learners struggling with geometry in the CVHPI group suggests that traditional methods of instruction are less effective for some learners. The fact that participants in the TVHPI group had this code indicates that learners who used this method had comparable difficulties, although less common than in the CVHPI group. Technical and resource-related difficulties could cause difficulties in understanding geometric ideas, since most of the learners in the TVHPI group reported this challenge compared to the CVHPI group. For example, a TVHPI participant said, "The computer was slow, and the power went out while the teacher was explaining rotations, so I still found matrix transformations and rotations difficult" (FLA/TVHPI/R1).

The differences in the technical and resource issues presented by the two instructional strategies highlight the direct impact that the mode of instruction has on the kinds of problems that learners encounter. Moreover, in some cases, learners shun away from seeking help when they encounter challenges. For example, one participant from the TVHPI group noted, "I got lost a lot and didn't ask for help enough" (FLA/TVHPI/U1), indicating that they had trouble asking for assistance during classes. Even while this problem was only occasionally mentioned, it brings to light a particular difficulty with technology use because learners can be less inclined to ask for help when using new digital tools. While not noticed in the CVHPI group, this issue highlights a possible disadvantage of technology-enhanced instruction: learners may experience a sense of isolation in their learning. This suggests that educators should create a setting where learners feel free to seek assistance, and encourage peer-to-peer interaction and group work, more than individualized learning.

The aforementioned difficulties highlight the necessity of metacognitive and self-regulated learning, which several learners still identified as areas of difficulty. These abilities (discussed in the following subsection) are essential for empowering learners to take charge of their education and identify when they need help, particularly in settings where technology or teaching strategies present extra challenges.

Metacognition and Self-Regulated Learning

Although fewer instances were recorded in the Metacognition and Self-Regulated Learning category (2 in TVHPI and 8 in CVHPI), this area shows how learners used self-regulation strategies to deal with their learning difficulties. These techniques played a crucial role in how learners overcame their learning challenges. The learners reported techniques such as *concentration, forging an interest in learning, and self-reflection after the lessons.*

Remarkably, a few participants in the TVHPI group (2 instances) stated that concentration or focus was necessary to get beyond obstacles when learning geometry. Learners could concentrate better on activities that improved their comprehension of the subject matter. "The shapes moved, but it did not always make sense to me. I think the learners who did better spent more time on the computer and paid more attention" (FLA/TVHPI/U1), a participant from the TVHPI group reported. This implies that technology can improve cognitive engagement, which helps learners understand complex geometric transformations more fully. It should be noted that this focus could be influenced by the learners' level of interest in what is being learned. Relatedly, two of the CVHPI group's participants stated that they overcame challenges because they were motivated to master geometry concepts.

Accordingly, learners in the CVHPI group more frequently mentioned self-reflection. One learner explained: "The teacher explained it, but sometimes I did not ask for more help, but after class, I would think about what we did, and that helped me understand better." (MLA/VHPI/R1). The reliance on self-reflection in CVHPI suggests that learners in traditional instructional settings often need to take more initiative in their learning process, potentially due to the lack of interactive feedback mechanisms available in technology-enhanced settings.

External Learning Support

The different types of help that learners thought they required to successfully navigate their learning experiences are included in this subsection. With a total of 46 instances (26 in TVHPI and 20 in CVHPI), it was evident that learners in both instructional modalities needed extra support to get beyond their obstacles. The categories of help that emerged from the participants included *extensive use of GeoGebra*, *Need for individualized support from the teacher, and allowed time for continuous practice.* Primarily, learners in the TVHPI group (six instances) underlined that knowledge of transformation geometry requires considerable use of GeoGebra. They admitted that even while the program made learning easier, proficiency still required a lot of practice as one of the participants stated: "More time on the computer would have helped" (FLA/TVHPI/R3). This emphasizes how crucial it is to practice technology on a regular and consistent basis to improve conceptual understanding

Moreover, this emphasizes the ongoing need for individualized support and increased practice time, since learners need both unique direction and chances to solidify their understanding of the subject matter. For example, with an equal frequency of 20 occurrences in each group, it was clear that both learners needed time for practice and specialized (external) support. A learner from the TVHPI group and CVHPI group respectively commented: “I think it would help if we had more time on the computer and more help from the teacher during the hard parts” (FLA/TVHPI/R1). "If the teacher gave me more time on that, it would have been better for me” (MHA/VHPI/U2). The fact that both groups have this desire emphasizes the shortcomings of both instructional strategies.

Instructional Effectiveness and Learner Satisfaction.

This theme looks at five emerging categories: beginning with the *Role of Instructional Strategies in Learning,* which forms the foundation for *Instructional Support and Guidance*. This support is then adapted through *Personalized Learning and Differentiated Instruction*, ensuring learners receive targeted help. Learners engage in Practical Learning and Reinforcement through this tailored approach, applying their knowledge in hands-on activities. Ultimately, the effectiveness of these methods is assessed against *Learners' Expectations*, evaluating how well the instructional strategies met their expected learning outcomes. These categories demonstrate the different experiences that the two groups had and show how each teaching strategy impacted learners' overall satisfaction and helped them learn. The occurrence of different codes in the data for each category and code is illustrated in Table 2.

Table 2

Showing a Count of Codes, and Categories (Theme 2)

Row Labels	TVHPI	CVHPI	Total
Instructional Effectiveness and Learner Satisfaction	**147**	**125**	**272**
Role of Instructional Strategy in Learning	**91**	**47**	**138**
Ability to Resolve misunderstanding	40	14	54
Enhanced Visual Learning	44	8	52
Step-by-Step Learning Approach	7	24	31
Instructional support and Guidance	**8**	**2**	**10**
Benefits of Multiple Explanations		2	2
Desire for More Time with GeoGebra	8		8
Personalized Learning and Differentiated Instruction	**2**	**6**	**8**
Enhanced Visual Learning	1		1
Importance of Prior Knowledge		5	5
Repeated Explanation and Personal Attention	1	2	3
Practical Learning and Reinforcement	**21**	**46**	**67**
Understanding through Examples	13	31	44
Improvement Through Practice	8	15	23
Learners expectations	**25**	**24**	**49**
The method didn’t meet most of the learner's expectations		13	13
The method met most of the learner's expectations	25	11	36

Role of Instructional Strategy in Learning

A total of 138 responses emerged in this category (91 in TVHPI and 47 in CVHPI); it is clear that the instructional strategy played a significant role in shaping learners' learning outcomes and their ability to engage with geometry concepts. The roles of instructional strategies that emerged under this category are the *ability to resolve misunderstandings*, *enhanced visual learning*, and a *step-by-step learning approach*. Learners in the TVHPI group overwhelmingly reported that the instructional strategy helped them resolve misunderstandings. A TVHPI learner shared: "Using GeoGebra helped me a lot because I could see the shapes change right in front of me. I understood what I had misunderstood" (FHA/TVHPI/R1).

In contrast, fewer learners in the CVHPI group mentioned resolving misunderstandings through the instructional strategy. One CVHPI learner said: "Reflections were easier for me, but when we moved on to rotations, they were confusing at first, but after following the steps I saw some light" (FLA/VHPI/R3). This finding suggests that the interactive and visual nature of TVHPI, particularly through the use of GeoGebra, allowed learners to identify and correct their mistakes more easily compared to CVHPI.

In support of this result, the role of visual learning was significantly more prominent in TVHPI, with 44 occurrences compared to eight in CVHPI. A TVHPI learner remarked: "GeoGebra made a big difference for me because I could see how the shapes were changing, (FHA/TVHPI/R1). Meanwhile, CVHPI learners noted the limitations of traditional visual aids, with one learner explaining: "I remember one lesson where the teacher showed us how to use symmetry, and that helped me" (FHA/VHPI/U3). The emphasis on visual learning in TVHPI highlights the advantage of technology in providing dynamic, real-time demonstrations of geometric concepts. The use of visual aids in TVHPI improved understanding through dynamic visualization; however, in CVHPI, the lack of visual aids necessitated a planned, systematic progression to lead learners through each concept and guarantee mastery at every level. The step-by-step learning approach was more commonly reported by CVHPI learners, with 24 occurrences compared to seven in TVHPI. The approach employed in CVHPI emphasizes the significance of Instructional Support and Guidance in fostering learner confidence through its organized and progressive approach. The continuous need for teacher support was evident in both CVHPI and TVHPI, as the former relied on direct teacher interaction and the latter needed explicit advice to supplement its visual aids.

Instructional Support and Guidance

Instructional support and guidance played a critical role in helping learners navigate their learning experiences. With 10 occurrences in this category (eight in TVHPI and two in CVHPI), the level of support provided was a key factor in shaping learners' perceptions of the effectiveness of their instructional strategy. This category was built on two codes: *desire for more time with GeoGebra and benefits of multiple explanations*.

To improve their understanding, learners in the TVHPI group frequently stated that they needed more time to spend with GeoGebra. This wish emphasizes how beneficial technology can be to education, especially when learners feel sufficiently supported. However, it also implies that the amount of time allotted to using these tools would not have been enough to achieve mastery, which might have reduced overall satisfaction with the teaching approach. For example, one learner remarked: "I think I needed more time with GeoGebra to understand" (MLA/TVHPI/R2). This result is consistent with the previous theme on Challenges and Support Needs, where learners stressed that to improve understanding, they needed to practice GeoGebra a lot.

While learners in the TVHPI group stated that more time spent using GeoGebra improved their comprehension, those in the CVHPI group highlighted that teacher explanations and multiple examples were crucial to their improvement, emphasizing the unique advantages of each instructional strategy in promoting learner learning. For example, a participant in the CVHPI group said, "The way the teacher taught us helped me get better because we practiced a lot" (MHA/VHPI/U2). Learners in this group also emphasized the advantages of hearing repeated explanations. Learners' overall pleasure with the method was probably influenced by the CVHPI instructor's ability to tackle a single concept from multiple perspectives. This is related to the notion that, although TVHPI is superior in visual and interactive learning, the instructor's flexibility in providing various forms of assistance makes CVHPI so strong. While both groups gained from their different teaching strategies, CVHPI learners from teacher explanations and various examples, and TVHPI learners from more time spent with GeoGebra, TVHPI had a greater capacity for encouraging independent learning.

Personalized Learning and Differentiated Instruction

This category highlights the importance of adapting instruction to meet the needs of individual learners. With eight occurrences overall (2 in TVHPI and 7 in CVHPI), it is clear that CVHPI provided more opportunities for personalized and differentiated learning. *Repeated explanations, personal attention, and the importance of prior knowledge* were the primary focus of this category. A CVHPI learner said that direct teacher intervention helped clear up uncertainty regarding rotation. On the other hand, it seems that CVHPI provides more regular chances for this customized advice. This result supports the idea of learner assistance more broadly by demonstrating that although technology like GeoGebra can enhance understanding, it cannot wholly replace one-on-one instructor interaction in helping learners learn complete ideas. This dependence on human attention and repeated explanation is also related to the importance of prior knowledge, since learners frequently need extra help to fill in the gaps between what they already know and new geometric concepts.

Practical Learning and Reinforcement

Practical learning and reinforcement were significant factors in shaping students' learning experiences and their overall satisfaction with the instructional strategies. With 67 occurrences (21 in TVHPI and 46 in CVHPI), both groups heavily emphasized this category, though it was more prominent in CVHPI. The two principal codes that emerged under this category were *the role of examples and practices in enhancing understanding*. Examples were critical to enhancing learners' understanding, with CVHPI learners reporting this benefit more frequently. The higher frequency of occurrences in CVHPI suggests that examples may have a more substantial impact on reinforcing understanding, possibly due to the nature of the instruction. However, the TVHPI method also proved effective in using examples, particularly through visual and interactive demonstrations provided by GeoGebra. This finding reinforces the idea that both instructional strategies offer valuable forms of reinforcement, though in different ways, and connect to the larger discussion of instructional support and guidance. All the instructional strategies were equally represented in the code "improvement through practice", with eight occurrences each. A learner from the TVHPI group remarked: ″I thought it would be confusing, but GeoGebra made everything clearer for me, especially with enlargements after practicing" (FHA/TVHPI/U3). Similarly, a CVHPI learner explained: ″I found reflections easier because we practiced them a lot" (MLA/VHPI/R3). The equal emphasis on practice across both instructional strategies emphasizes the importance of consistent reinforcement in mastering geometry concepts. Whether through digital tools or traditional exercises, learners recognized that repeated practice was key to their improvement.

Learners' Expectations

This examines how the instructional strategies and learners' expectations for learning geometry correspond. Throughout 49 instances (25 in TVHPI and 24 in CVHPI), learners from the two groups expressed differing opinions about how well the instructional strategies fulfilled their expectations for their learning. The surfaced codes represent the different experiences and satisfaction with the instruction strategies, including *the Method Met Most of the Learners' Expectations, and the Method Did Not Meet Most of the Learners' Expectations.*

All 25 mentions from the TVHPI group stressed that the instructional strategy met the learners' expectations compared to 13 out of 24 from the CVHPI. For example, one TVHPI learner explained: ″I did not think I would enjoy using the computer for geometry, but I was surprised at how much fun it was" FHA/TVHPI/R2).

Meanwhile, a CVHPI learner noted: ″I thought reflections would be the most confusing part, but I was surprised at how fast I picked them up" (FHA/VHPI/U1). The higher satisfaction reported by TVHPI learners suggests that the use of technology supported learners' understanding and aligned more closely with their expectations for a modern, interactive learning experience. This finding connects back to the earlier codes related to visual learning and the ability to resolve misunderstandings, which likely contributed to learners feeling that their expectations were met. However, the CVHPI method, while effective in many ways, may have fallen short for learners seeking more dynamic or interactive elements in their learning experience. However, even if CVHPI was effective in many ways, learners who had hoped for a more engaged or participatory approach might not have been delighted. This led to multiple reports from the CVHPI group saying that the procedure fell short of what they had anticipated. One CVHPI learner said, ″I did not think I would be so confused with symmetry. I expected it to be simpler, but I kept making mistakes when it came to figuring out where the lines went. It surprised me how much more attention I needed to pay" (FLA/VHPI/U1). This narrative reveals that learners in CVHPI did not receive the expected experiences and understanding of transformation geometry, resonating from the challenges faced.

Discussion of the Findings

This discussion addresses the research questions by analyzing the challenges and support needs associated with TVHPI and CVHPI, as well as learners' perceptions of the effectiveness of these methods in enhancing their understanding of transformation geometry.

Challenges and Support Needs Associated with TVHPI and CVHPI

The findings reveal that both TVHPI and CVHPI present unique challenges, though the nature of these challenges varies. Consistent with prior research, learners in the CVHPI group reported struggling more with conceptual understanding, particularly for abstract transformations like rotations and scaling. This aligns with studies emphasizing the limitations of traditional instruction in fostering geometric reasoning without dynamic visual aids (Açıkgül, 2022). In contrast, TVHPI learners highlighted challenges related to technical and resource constraints, such as unreliable electricity and insufficient exposure to GeoGebra, supporting literature that emphasizes the need for robust infrastructure in technology-enhanced learning (Adelabu et al., 2022; Afshari et al., 2009; Roxana, 2019). This is consistent with studies on the digital divide, which argue that disparities in access to technology can exacerbate educational inequalities (Moore et al., 2018). While TVHPI offers significant pedagogical advantages, its success is contingent on addressing these infrastructural challenges, particularly in under-resourced contexts.

However, learners in the TVHPI group frequently mentioned the benefits of enhanced visual learning. The ability to manipulate geometric shapes through GeoGebra allowed learners to move through the Van Hiele levels more effectively, as they could simultaneously engage with the conceptual and procedural aspects of transformations. This aligns with cognitive load theory (Plass et al., 2010), which suggests that reducing the cognitive demands associated with abstract problem-solving tasks (such as visualizing geometric transformations) allows learners to focus more on understanding the underlying principles.

Consequently, learners in both groups expressed a need for individualized support and extended practice, emphasizing the universal role of teacher scaffolding and repetition in mastering geometry concepts. Interestingly, TVHPI learners noted a reluctance to seek help, potentially due to a sense of isolation when engaging with technology. This finding contrasts with studies suggesting technology fosters collaboration and engagement (Zheng et al., 2022), highlighting the importance of creating supportive environments in technology-enhanced instruction.

Learners' Perception of Effectiveness of TVHPI and CVHPI in Enhancing Understanding

Regarding perceptions of instructional effectiveness, TVHPI was consistently praised for its ability to enhance conceptual understanding through dynamic visualization. Learners frequently noted that GeoGebra allowed them to "see" transformations in real-time, facilitating deeper comprehension of abstract concepts like rotations and reflections. This aligns with Sunzuma (2023) and Wachira and Keengwe (2011) findings on the power of technology to make abstract geometric concepts more accessible. In contrast, CVHPI learners benefited more from the structured nature of conventional instruction, which helped reinforce procedural mastery. This is consistent with research suggesting that conventional methods remain effective for learners who rely heavily on explicit guidance and the fact that the method uses the structured way of teaching emphasized by Van Hiele (Bonyah & Larbi, 2021; Hattie, 2009; Machisi & Feza, 2021; Moru et al., 2021; Tahani, 2016).

However, satisfaction levels differed between the groups. TVHPI learners expressed greater alignment between their expectations and instructional outcomes, likely due to the engaging and interactive nature of the technology. This finding supports literature highlighting the growing preference for technology in modern classrooms (Adelabu et al., 2022; Afshari et al., 2009). Conversely, CVHPI learners valued the multiple explanations and personalized attention provided by their teachers, suggesting that conventional methods retain strengths in areas where technology cannot fully replicate human interaction (Ardeleanu, 2019; Lessani et al., 2017; Roxana, 2019; Tularam, 2018).

Despite these positive perceptions, both groups emphasized the importance of external support, particularly through teacher guidance and extended practice opportunities. The shared reliance on teacher intervention underscores the complementarity of instructional strategies, with technology enhancing conceptual understanding and conventional methods reinforcing procedural skills.

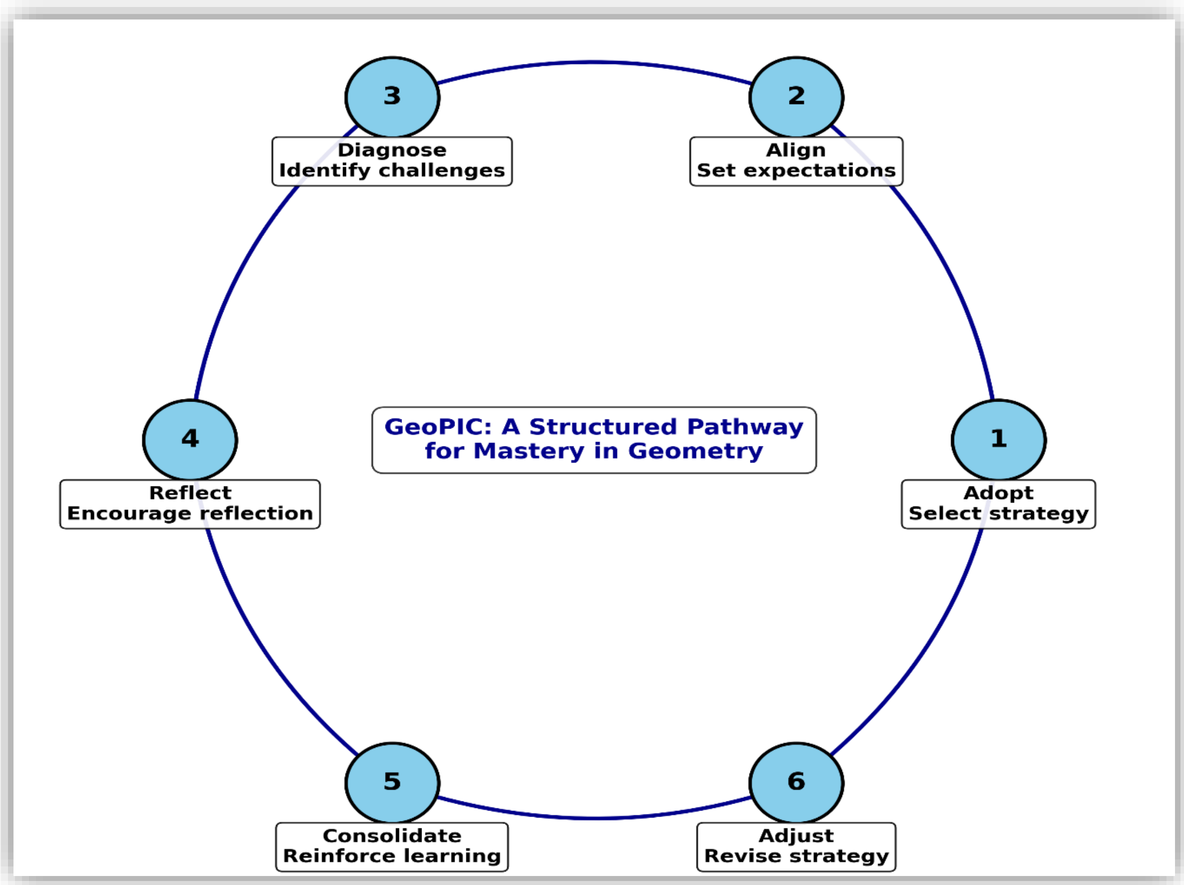
Synthesis of Findings

The findings reveal that while TVHPI addresses conceptual challenges effectively, it introduces technical barriers that require careful planning and support. Conversely, CVHPI fosters procedural understanding and self-regulation but may fall short of meeting learners' expectations and seeking interactive experiences. These insights suggest that neither method is inherently superior. Instead, their effectiveness depends on aligning instructional strategies with learners' needs and preferences. As a result, the study suggests a Geometry Pedagogical Improvement Cycle (GeoPIC-Framework) that provides a structured pathway for learners to advance from basic shape recognition to abstract geometric reasoning, ensuring a seamless transition through Van Hiele's levels of geometric thinking. This cycle is illustrated in Figure 4.

The cycle begins with adopting an instructional strategy that aligns with the learners' abilities and learning objectives/outcomes. Teachers must determine whether to employ traditional hands-on tools like graph paper, mirrors, and rulers or to enhance learning through technology using dynamic visualization platforms such as GeoGebra. Once a strategy is in place, the focus shifts to aligning expectations, where learners are introduced to the structure of their learning journey; understanding what they will learn, how they will engage with geometric transformations, and how their progress will be assessed. This alignment ensures that learners are not merely following procedures but actively constructing their understanding of geometric concepts.

Figure 4

Showing the Geometry Pedagogical Improvement Cycle (GeoPIC-Framework)



With expectations set, the next critical step is diagnosing learning challenges to establish each learner's entry point in the learning process. Through pre-tests, discussions, and observation of learner interactions, educators identify misconceptions and conceptual gaps that may hinder progress. This diagnosis is essential in structuring subsequent instruction to address weaknesses while reinforcing foundational geometric principles. Once challenges are identified, the cycle moves to reflection, where learners and teachers critically analyze learning experiences. Learners engage in self-explanation and problem analysis, strengthening their metacognitive awareness and helping them transition from visual recognition to logical deduction. Meanwhile, teachers assess the effectiveness of their instructional approach, refining strategies to improve conceptual clarity and engagement.

Following reflection, learning is consolidated through practice and application. Learners apply their understanding in guided activities, structured problem-solving tasks, and real-world scenarios using manual drawing techniques and interactive digital explorations. Technology, particularly GeoGebra, is crucial in reinforcing geometric transformations by allowing learners to manipulate figures dynamically, observe patterns, and test conjectures. This phase cements knowledge, enabling learners to analyze and apply transformations confidently rather than memorize rules.

However, learning is not static. As learners engage in practice, adjustments must be made to enhance instruction further. Teachers evaluate progress and refine their approaches, modifying teaching pace, instructional materials, and learner support strategies to better align with evolving needs. This continuous refinement ensures that learners do not stagnate but advance systematically through increasingly complex geometric reasoning tasks.

The GeoPIC framework is not a linear process but an ongoing cycle, looping back to adoption and realignment as new insights emerge. Each iteration strengthens learners' understanding, moving them from basic spatial visualization to higher-order geometric reasoning. By integrating traditional instructional techniques with dynamic digital tools, this framework provides an adaptive, research-driven approach to teaching geometry, ensuring that learners not only master transformations but also develop the ability to think, reason, and engage with geometric concepts at a deeper level. The iterative nature of this cycle ensures that learning is continuous, responsive, and progressively builds toward geometric mastery, equipping learners with the essential skills to analyze spatial relationships, apply logical reasoning, and connect geometric principles to real-world contexts.

The continuous feedback loop also ensures improvement in teaching effectiveness and student learning outcomes. GeoPIC is based on key educational theories, including the Van Hiele Theory of Geometric Thinking, Differentiated Instruction, and constructivist principles. It emphasizes tailoring strategies to learners' needs, active learning, and continuous reflection; Crowley, 1987; Vojkuvkova, 2012). Formative assessment guides the cycle's feedback stages, while its cyclical process mirrors the Plan-Do-Check-Act model for continuous teaching improvement (Pratik & Vivek, 2017).

Limitations of the Study

The study's sample size (48 Senior Three learners from six Ugandan schools) limits generalizability, as findings may not fully apply to other grade levels. While Senior Three was chosen due to its alignment with transformation geometry content, future studies could explore the effectiveness of TVHPI in other classes to assess broader applicability. Additionally, although teachers were trained and standardized lesson plans were developed, variations in teaching styles and instructional delivery were inevitable and may have influenced some results. Despite these limitations, the study's qualitative depth, diverse school selection, and rigorous methodology ensure credible insights.

Conclusions and Recommendations

The study identified challenges and support needs associated with TVHPI, particularly its reliance on digital tools, which posed obstacles such as electricity instability and limited device access in low-resource settings. Since CVHPI is embedded within TVHPI, it remains a viable alternative when technology is unavailable, ensuring learners experience Van Hiele Phased Instruction through traditional tools. To address these challenges, a hybrid approach integrating offline GeoGebra use and paper-based simulations could enhance accessibility. Additionally, while TVHPI promotes individual exploration, some learners reported feeling isolated compared to the collaborative nature of CVHPI. Peer-to-peer interactions and structured group tasks within TVHPI would help maintain learner engagement while maximizing technology's benefits.

Regarding learners' perceptions of effectiveness, TVHPI enhanced understanding of transformation geometry by providing dynamic visualization, especially for rotations and reflections, allowing learners to manipulate shapes and observe transformations in real time. Within TVHPI, CVHPI remained essential for structured reinforcement, supporting procedural accuracy through manual plotting, mirrors, and graph paper. While TVHPI strengthened conceptual understanding, CVHPI ensured learners developed procedural fluency, highlighting the need for a balanced integration of both approaches to optimize learning in transformation geometry.

Schools should prioritize TVHPI as the primary instructional strategy, leveraging its interactive and visual capabilities while retaining CVHPI's structured support for personalized learning. Teachers should receive comprehensive training on GeoGebra and digital tools, integrated into continuous professional development with hands-on practice. Addressing technical challenges, such as power reliability and device access, is crucial for seamless integration. GeoGebra can be used as a dynamic board to enhance visualization where devices are limited. A hybrid approach would optimize learner learning by combining TVHPI's visualization strengths with CVHPI's structured guidance. The GeoPIC framework should guide continuous instructional improvements.

At the policy level, teacher training should transition from CVHPI to TVHPI, ensuring educators develop strong foundational skills before integrating technology. Policymakers should invest in affordable digital solutions and offline learning resources, expanding access in low-resource settings. Hybrid models integrating technology and traditional tools should be supported to ensure inclusive learning environments.

Further research should explore TVHPI's effectiveness across more extensive and diverse samples. Additionally, the practicality of the GeoPIC framework should be assessed to refine instructional strategies. Longitudinal studies should evaluate knowledge retention and higher-order reasoning skills, ensuring sustainable benefits of TVHPI in mathematics education.

Contribution: By examining the Van Hiele Phased Instruction (VHPI) model in Uganda and investigating the effectiveness of Technology-Enhanced Van Hiele Phased Instruction (TVHPI) with the integration of GeoGebra, this study enhances the instruction of mathematics. This research sheds light on how different approaches to instruction affect the learner's understanding of transformation geometry and tries to fill the gap in the literature. In addition to improving geometry instruction, this study presents the Geometry Pedagogical Improvement Cycle (GeoPIC), a framework that can address learner obstacles and improve teaching strategies.

Data Availability: The data for this study can be accessed through this link:
https://docs.google.com/spreadsheets/d/1aIfYwD3LX1SBOGRpqPq-22-f-On0BuGD/edit?gid=1200165066#gid=1200165066

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A Qualitative Document Analysis of Quantitative and Covariational Reasoning Opportunities Provided by Calculus Textbooks: The Case of Related Rates Problems

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ABSTRACT

With a focus on related rates problems, the present study reports on quantitative and covariational reasoning opportunities provided by five widely used calculus textbooks in the United States. There are three major results from this study. First, quantitative reasoning opportunities are plentiful, while covariational reasoning opportunities are scarce in all the textbooks, respectively. Second, there is a severe shortage of related rates problems that require more than recalling geometric formulas to mathematize. Third, opportunities promoting the use of diagrams to support students' quantitative reasoning when solving related rates problems are minimal in the practice problems provided in the five textbooks. Overall, the textbooks provide limited opportunities to engage in covariational reasoning when working with related rates problems. Implications for instruction are discussed.

Keywords: Quantitative reasoning, covariational reasoning, related rates problems, derivatives, calculus, document analysis, opportunity to learn

Introduction

A growing number of scholars have called for helping students develop strong quantitative and covariational reasoning abilities, respectively, arguing that this is necessary for students to acquire robust understandings of mathematical concepts/topics that involve making sense of quantities and how these quantities change in relation to each other such as related rates problems in calculus (e.g., Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Moore, 2014; Thompson, 1994, 2011). I remark that related rates problems form an integral part of any first-semester calculus course in the United States (e.g., Engelke, 2007, Engelke-Infante, 2021; Mkhatshwa, 2020a).

A mathematical task is a related rates problem if it involves at least two 'rate' quantities that can be related by an equation, function, or formula (Mkhatshwa, 2020a). There are two types of related rates problems, namely geometric and non-geometric. According to Mkhatshwa (2020a), "a geometric related rates problem is one in which the equation relating the quantities [in the problem] is based on a geometric structure such as the Pythagorean Theorem or the volume of a shape" (p.141). Analogously, a non-geometric related rates problem is one in which the equation relating the quantities in the problem is based on a non-geometric relationship such as some Physics laws (e.g., the ideal gas law) or the economics formula $P = R - C,$ where P is profit, R is revenue, and C is cost.

A historical analysis of related rates problems by Austin et al. (2000) reveals that the inclusion of related rates problems in calculus textbooks dates back to at least 1836. Furthermore, these authors found that related rates problems first appeared in the United States in 1851 in a calculus textbook published by Elias Loomis (1811-1889), who was a mathematics professor at Yale University. Students' difficulties with related rates problems, including those directly related to quantitative or covariational reasoning, respectively, are widespread and have continued to be a subject of empirical research investigations for the last 25 years (e.g., Alvine et al., 2007; Azzam et al., 2019; Code et al., 2014; Ellis et al., 2015; Engelke, 2007; Engelke-Infante, 2021; Jeppson, 2019; Kottath, 2021; Martin, 2000; Mkhatshwa & Jones, 2018; Mkhatshwa, 2020a, 2020b; Picollo & Code, 2013; Taylor, 2014; White & Mitchelmore, 1996). In fact, within the last few years alone, several different studies (e.g., Azzam et al., 2019; Engelke-Infante, 2021; Jeppson, 2019; Kottath, 2021; Mirin & Zaskis, 2019; Mkhatshwa, 2020a) have reported on students' difficulties in connection with related rates problems. Evidence from a related line of research suggests the existence of a correlation between learning opportunities provided by mathematics textbooks and difficulties exhibited by students in formal assessments (e.g., Schmidt et al., 2015).

There are generally three different flavors of first-semester calculus offered at the undergraduate level in the United States, namely regular calculus (also known as engineering calculus), life sciences calculus, and business calculus. I note that while related rates problems are a common topic in regular calculus textbooks, they are not covered in most life sciences or business calculus textbooks. The five textbooks considered in this study include three regular calculus textbooks (Stewart et al., 2021; Hughes-Hallett et al., 2021; Rogawski et al., 2019), one life sciences calculus textbook (Greenwell et al., 2015), and one business calculus textbook (Barnett et al., 2019). The research question guiding this study is: What opportunities do calculus textbooks offer students to engage in quantitative reasoning or covariational reasoning when solving related rates problems? I remark that the purpose of this paper is not to make a theoretical contribution, but rather to describe learning opportunities in the context of quantitative reasoning and covariational reasoning provided by calculus textbooks. Additionally, the present study uses the term real-world context broadly to either refer to a relevant and essential context or a camouflage context (e.g., Wijaya et al., 2015). It is worth noting that tasks with the former type of real-world context typically provide more opportunities to engage in quantitative reasoning compared to tasks that have the latter type of real-world context (e.g., Vos, 2020).

Background for the Study

In a recent study (Mkhatshwa, 2022), I reported on quantitative and covariational reasoning opportunities provided by two widely used calculus textbooks in the United States. The focus of the study was on ordinary derivatives and partial derivatives. A key finding of the recent study is that there is a dearth of opportunities to engage in covariational reasoning in connection with ordinary or partial derivatives. Furthermore, the study found that while opportunities to engage in quantitative reasoning are prevalent in one of the textbooks (an applied calculus textbook), there is a short supply of similar opportunities in the other textbook (a traditional calculus textbook).

To ascertain whether the findings of the recent study could be generalized to other topics covered in widely used calculus textbooks in the United States, the present study reports on opportunities to engage in quantitative reasoning and covariational reasoning, in the context of related rates problems, provided by five commonly used calculus textbooks in the United States. I note that two of the five textbooks in the present study were examined in the recent study on ordinary derivatives and partial derivatives. In essence, the present study examined a different topic (related rates problems) compared to the recent study.

Moreover, the present study examined three more textbooks compared to the recent study. As I show in the results section, findings (especially concerning covariational reasoning opportunities) of the present study are very similar to findings of the recent study. The observed similarities in the findings of the two studies lead me to the conclusion that there is generally a paucity of covariational reasoning opportunities in calculus textbooks used in the United States. Arguably, findings of both studies extend beyond the United States, as some of the widely used calculus textbooks in the United States are also used in other countries such as Canada. I therefore argue that calculus textbook authors should strive to include these seemingly lacking opportunities in future editions of their textbooks, in light of the significant role textbooks play in students' learning of mathematics, among other things.

Related Literature

Opportunity to Learn

Although there are slight variations in how the concept of opportunity to learn has been defined in the mathematics education literature, this concept has been used in the same literature for over half a century. For instance, Carrol (1963) defined opportunity to learn as the amount of time devoted to learning about a particular topic, while Husén (1997) defined the same concept as whether or not "students have had the opportunity to study a particular topic or learn how to solve a particular type of problem" (pp. 162-163). According to Floden (2002), Husen's definition of opportunity to learn is commonly used in the mathematics education literature.

This study uses Husén's (1997) definition of opportunity to learn. Specifically, it examined whether or not widely used calculus textbooks in the United States provide opportunities for students to engage in quantitative reasoning or covariational reasoning in the context of working with related rates problems. This examination is particularly important because evidence from a related line of research suggests that student achievement in particular areas/topics of study is tied to the extent to which they have had an opportunity to learn about these areas/topics, such as via classroom instruction or course textbooks (e.g., Cogan & Schmidt, 2015). The significance of textbooks in students' learning of mathematics cannot be overstated. In fact, according to Reys et al. (2004), "the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61), a sentiment that has been echoed by other researchers (e.g., Alajmi, 2012; Kolovou et al., 2009). In this study, the terms "opportunity to learn" and "learning opportunities" are used interchangeably.

An Overview of Mathematics Textbook Research at the K-12 and University Level

Research on learning opportunities provided by mathematics textbooks at the K-12 level (i.e., from Kindergarten to Grade 12) has not only received substantial attention, but also covers a wide range of topics, including cognitive demands of mathematical tasks (e.g., Basyal et al., 2022; Gracin, 2018), deductive reasoning (e.g., Stacey & Vincent, 2009), fractions (e.g., Alajmi, 2012; Charalambous et al., 2010), functions (e.g., Wijaya et al., 2015), problem solving (e.g., Jäder et al., 2020), mathematical reasoning and proof (e.g., Stylianides, 2009; Thompson et al., 2012), probability (e.g., Jones & Tarr, 2007), proportional reasoning (e.g., Dole & Shield, 2008), statistics (e.g., Pickle, 2012), trigonometry (e.g., Wijaya et al., 2015), and students' perceptions regarding the role of textbooks in their learning of mathematics (e.g., Wang & Fan, 2021).

On the contrary, similar research at the undergraduate level has not received considerable attention. The focus of the available studies at the undergraduate level has mainly been on cognitive demands of tasks typically found in mathematics textbooks (e.g., Mesa et al., 2012), learning opportunities related to the concept of the derivative (e.g., Haghjoo et al., 2023, Park, 2016), continuity (e.g., Raman, 2004), optimization problems(e.g., Mkhatshwa & Doerr, 2016; Mkhatshwa, 2023), infinite series (e.g., González-Martín et al., 2011; Heon & Mills, 2023; O'Sullivan et al., 2023), the usage of multiple ways (i.e., algebraically, graphically, numerically, or verbally) to represent mathematical ideas such as functions (e.g., Chang et al., 2016), and limits (e.g., Hong, 2022; Lithner, 2004). On a related note, González-Martín et al. (2018) reported on a case study of how five instructors use a common textbook to prepare for teaching series in calculus. Mesa and Griffiths (2012) described three ways course textbooks mediate the work of college faculty, namely "textbook mediation between instructor and design of instruction" (p. 93), "textbook mediation between instructor and others" (p. 95), and "textbook mediation between instructor and self" (p. 98). According to Mesa and Griffith (2012):

> Reflexive mediation between the textbook and instructors manifests when instructors make mental or physical notes about things that work or do not work, find examples or problems that they need to modify or remove, and identify topics they will not cover or will cover next time they teach (p. 98).

In the context of opportunity to learn, textbook mediation between instructor and self could, for instance, manifest when instructors find or modify examples or problems to supplement essential learning opportunities that are lacking or minimal in the textbooks adopted for their courses. It is worth mentioning that most of the participants in Mesa and Griffiths' (2012) study were calculus instructors. Randahl (2012) reported on how first-year engineering students use mathematics textbooks in their learning of calculus.

The Significance of Textbooks in Mathematics Education

Textbooks play a crucial role in students' learning of mathematics. A recurrent finding from research that has scrutinized the significance of textbooks in the teaching and learning of mathematics at all levels is that nearly all mathematics content covered during classroom instruction is generally dictated by course textbooks (e.g., Begle, 1973; Rezat, 2006; Reys et al., 2004; Robitaille & Travers, 1992; Törnroos, 2005; Wijaya et al., 2015). Indeed, in an attempt to underscore the importance of mathematics textbooks, Begle (1973) asserted that most of what students learn is directed by textbooks rather than teachers. Similar assertions have been echoed by other researchers (e.g., Blazar et al., 2020; Polikoff, 2018; Polikoff et al., 2021).

Students' Difficulties with Engaging in Quantitative Reasoning or Covariational Reasoning when Solving Related Rates Problems

Several studies have reported that related rates problems have a reputation, among students, of being difficult to master (e.g., Alvine et al., 2007; Ellis et al., 2015; Engelke-Infante, 2021). A common finding of research that has examined students' reasoning in the context of working with related rates problems is that students often exhibit difficulties engaging in certain aspects of quantitative reasoning. In particular, a growing number of studies have reported on students who struggled with determining correct units of measure for quantities (e.g., Azzam et al., 2019; Mkhatshwa, 2020a, Kottath, 2021). White and Mitchelmore (1996) reported on students who treated variables representing quantities as symbols that are to be manipulated algebraically and not as quantities that are to be related.

Several studies that have investigated students' thinking about geometric-related rates problems have found that mathematizing (Freudenthal, 1993) this type of problem is problematic for students (e.g., Azzam et al., 2019; Jeppson, 2019; Martin, 2000; Mkhatshwa, 2020a; White & Mitchelmore, 1996). To be specific, mathematizing a related rates problem entails using algebraic symbols to represent the different quantities in the problem, in addition to using an equation/formula to relate the quantities. On a positive note, findings of recent studies on related rates problems suggest that using diagrams to support students' quantitative reasoning is effective when solving related rates problems (e.g., Engelke-Infante, 2021; Mkhatshwa, 2020a).

Evidence from research shows that students exhibit weak covariational reasoning abilities when solving related rates problems (e.g., Carlson et al., 2002; Engelke, 2007). Specifically, this research shows that students seldom engage in the highest levels of covariational reasoning when solving related rates problems. Findings from a related line of research on students' thinking about ordinary derivatives, crucial elements of any related rates problem in calculus, indicate that students' weak covariational reasoning abilities are often evident when they are engaged in solving application problems that involve working with quantities that can be represented using ordinary derivatives (e.g., Jones, 2017; Nagle et al., 2013).

Document Analysis and its Usefulness in Qualitative Research

Document analysis is a useful method in qualitative research (e.g., Merriam & Tisdell, 2016; Morgan, 2022). This study uses the definition of document analysis proposed by Bowen (2009):

> Document analysis is a systematic procedure for reviewing or evaluating documents–both printed and electronic (computer-based and Internet-transmitted) material. Like other analytical methods in qualitative research, document analysis requires that data be examined and interpreted in order to elicit meaning, gain understanding, and develop empirical knowledge [e.g., Corbin & Strauss, 2008; Rapley, 2007]. Documents contain text (words) and images that have been recorded without a researcher's intervention (p. 27).

The documents considered in the present study are five textbooks that are widely used in the teaching of undergraduate calculus in the United States. Factors to consider when selecting documents for analysis include authenticity, credibility, representativeness, and meaning (e.g., Morgan, 2022). According to Morgan (2022), authenticity examines the degree to which a document is genuine, credibility examines the accuracy of a document, representativeness examines the degree to which a document is typical, and meaning examines the degree to which a document's content is significant, clear, or understandable.

I conclude this section by highlighting a few reasons behind my choice of using document analysis in the present study. First, "information and insights derived from documents can be valuable additions to a knowledge base" (Bowen, 2009, p. 30). Second, document analysis does not involve collecting new data. Consequently, the resources (e.g., time and costs) associated with using this methodology are often minimal (Pershing, 2002).

Third, "document analysis can serve as either a stand-alone data-collection procedure or as a precursor to collecting new data using other methodologies" (Pershing, 2002, p. 36). I remark that in the present study, document analysis serves as a stand-alone data-collection procedure. Fourth, the documents are readily available in the public domain (Bowen, 2009). Fifth, document analysis is not affected by obtrusiveness and reactivity i.e. the documents are not affected by the research process (Bowen, 20009). As with any research methodology, document analysis has its own limitations. These include insufficient detail [i.e. documents are not often produced with a research agenda], low retrievability, and biased selectivity [of the documents to be analyzed]. Citing the efficiency and cost-effectiveness of document analysis, Bowen (2009) argued that the benefits of this method far outweigh its limitations.

Theoretical Perspective

Quantitative Reasoning and Covariational Reasoning

Developed nearly three decades ago, the theoretical constructs of quantitative reasoning and covariational reasoning are well known among most mathematics education researchers and practitioners (e.g., Carlson et al., 2002; Smith III & Thompson, 2007; Thompson, 1993, 2011). Consequently, this section provides a synopsis of these theoretical constructs in connection with the present study. The interested reader is referred to my recent study (Mkhatshwa, 2022) for a comprehensive description of the aforementioned theoretical constructs as they relate to the analysis of learning opportunities provided by mathematics textbooks. When measured, quantities have units of measure (e.g., Thompson, 1993). The length of a ladder, the radius of a snowball, and the distance travelled by a car are a few of many examples of quantities referred to in the present study. Quantitative reasoning entails quantification (i.e., determining numeric values for quantities), interpreting quantities, analyzing and determining units of measure for quantities, and analyzing quantities and relationships among quantities based on textual descriptions of problem statements, algebraic equations, graphs/diagrams, or numerical tables of values, respectively, among other things.

Covariational reasoning, on the other hand, deals with analyzing how two or more quantities are changing in relation to each other. Figure 1 provides a description, using the Ladder Problem as an example, of the five levels of covariational reasoning.

Figure 1

Ladder Problem (Reproduced from Carlson et al., 2002, p. 371)

> **From a vertical position against a wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.**

Coordination: At the coordination level of covariational reasoning, also known as Level 1, a recognition that two quantities are changing simultaneously is made. In terms of the related rates problem described in Figure 1, this could mean recognizing that the vertical distance and the horizontal distance are changing simultaneously as the bottom of the ladder is pulled away.

Direction: At the direction level of covariational reasoning, also known as Level 2, attention is given to how two quantities are changing (direction-wise) in relation to each other. This could mean recognizing that as the bottom of the ladder is pulled away, the horizontal distance increases while the vertical distance decreases.

Quantitative Coordination: At the quantitative coordination level of covariational reasoning, also known as Level 3, one coordinates the amount of change of at least one of the two quantities. A qualifying remark at this level could be something like the following: "The vertical distance decreases by 0.5 feet as the horizontal distance increases."

Average Rate: At the average rate of change level of covariational reasoning, also known as Level 4, the focus is on coordinating the average rate of change of one of the quantities with constant changes in the other quantity. A qualifying remark at this level could be a comment like the following: "The vertical distance decreases by 0.75 feet every time the horizontal distance increases by one foot."

Instantaneous Rate: At the instantaneous rate level of covariational reasoning, also known as Level 5, the focus is on coordinating the instantaneous rate of change of one of the quantities with continuous changes in the other quantity. That is, a person reasoning at the instantaneous rate level continuously quantifies how the vertical distance changes with much smaller (less than one foot) changes in the horizontal distance.

The Role of Quantitative Reasoning and Covariational Reasoning in Related Rates Problems

The combination of quantitative and covariational reasoning is crucial in making sense of related rates problems (e.g., Engelke, 2004; Mkhatshwa, 2020a). A multitude of physical or dynamic situations/events can be modelled using related rates problems in different disciplines, including physics, engineering, and economics. In physics, for example, the relationship between the height of a rocket that rises vertically and the angle of a camera placed several yards from the launch pad of the rocket can be modeled using a related rates problem. According to Engelke (2007):

> Solving a related rates problem requires that the student engage in covariational reasoning to understand how the problem works, construct a mental model that allows them to recognize which variables are changing, construct a meaningful relationship between the changing quantities (create an appropriate formula), and reconceptualize the variables in their formula as functions of time. Only then may they use the chain rule to correctly differentiate their formula with respect to time and solve for the desired variable. (p. 29)

Quantitative reasoning plays an important role in the process of solving any related rates problem that has a real-world context. Among other things, the final step when constructing a solution to a related rates problem involves engaging in the process of quantification (i.e., assigning a numerical value to the quantity described by Engelke (2007) as the "desired variable" in the preceding quotation). In fact, some of the previously reported challenges exhibited by students when tasked with solving related rates problems deal directly with quantitative reasoning. Mkhatshwa (2020a) theorized that while covariational reasoning is certainly a key construct when dealing with related rates problems, there may be quantitative ideas, such as the role and use of diagrams to represent relationships between quantities, at play.

As previously noted, there is a relationship between the opportunity to learn about a particular area/topic and students' achievement when assessed in the same area/topic (e.g., Cogan & Schmidt, 2015). Furthermore, both quantitative reasoning and covariational reasoning are essential for students hoping to develop a solid understanding of various calculus ideas, such as the concept of the derivative (e.g., Carlson et al., 2002; Mkhatshwa, 2024). Additionally, covariational reasoning is an essential mode of reasoning for students hoping to make sense of related rates problems in calculus (e.g., Engelke, 2007). According to Jones (2017), Carlson's five levels of covariational reasoning are increasingly sophisticated. Consequently, I posit that students who are able to engage at the highest levels of covariational reasoning demonstrate deeper levels of learning or understanding. Indeed, Carlson et al. (2002) reported on two students (Student A and Student B) who reasoned at the highest levels of covariational reasoning. The students' reasoning at the highest levels of covariational reasoning correlated with high achievement in a related rates task involving a spherically-shaped bottle that was filled with water. In light of the multitude of benefits associated with engaging in quantitative reasoning or covariational reasoning in the study of calculus, it is paramount that calculus textbook authors provide plenty of opportunities (e.g., expository sections, examples, and practice problems) for students to engage in the aforementioned modes of reasoning.

Methods

Analyzed Textbooks

Five textbooks commonly used in the teaching of regular calculus, life sciences calculus, and business calculus in the United States, respectively, were analyzed in this study. See Table 1 for information on the textbooks included for analysis in this study.

Table 1

Analyzed Textbooks

Textbook Name	Author(s)	Sections Analyzed	Textbook Publisher
Calculus: Early Transcendentals (9th ed)	Stewart et al. (2021)	3.9: Related Rates	Cengage Learning
Calculus: Early Transcendentals (4th ed)	Rogawski et al. (2019)	3.10: Related Rates	Macmillan Learning
Single Variable Calculus (8th ed)	Hughes-Hallett et al. (2021)	4.6: Rates and Related Rates	Wiley
Calculus for the Life Sciences (2nd ed)	Greenwell et al. (2015)	6.4: Related Rates	Pearson Education
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	Barnett et al. (2019)	3.7: Related Rates	Pearson Education

Before selecting these textbooks, I consulted with major textbook publishing companies in the United States, including Cengage Learning, Pearson Education, and Wiley, regarding commonly used or ordered calculus textbooks.

Regular calculus in the United States undergraduate mathematics curriculum is generally taken by Science, Technology, Engineering, and Mathematics majors, respectively. Life sciences calculus is typically taken by biology majors, while business calculus is mostly taken by business or economics majors, respectively.

Data Analysis

There are three sources of data for this study, namely (1) expository sections on related rates problems, (2) examples on related rates problems, and (3) practice problems listed at the end of the sections noted in Tables 1 and 2.

Expository sections, examples, and practice problems were analyzed through the theoretical constructs of quantitative reasoning and covariational reasoning, both of which are described in the theoretical perspective section. Additionally, I examined definitions of related rates problems as well as strategies for solving related rates problems as part of my analysis of expository sections. Furthermore, examples or practice problems (hereafter, tasks) were classified as either having real-world contexts or mathematics contexts. In my recent study (Mkhatshwa, 2022), I explained that tasks with the former type of contexts provide opportunities to engage in quantitative reasoning while tasks with the latter type of contexts do not.

Table 2

Counts of Examples, Practice Problems, and Expository Sections

Textbook Name	Section	Expository Sections	Examples	Practice Problems
Calculus: Early Transcendentals (9th ed)	3.9	2	5	53
Calculus: Early Transcendentals (4th ed)	3.10	2	5	45
Single Variable Calculus (8th ed)	4.6	1	4	69
Calculus for the Life Sciences (2nd ed)	6.4	2	6	36
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	3.7	2	4	48
Total		**9**	**24**	**251**

Lastly, evidence from research on students' thinking about related rates problems suggests that students struggle with mathematizing related rates problems (e.g., Azzam et al., 2019; Jeppson, 2019; Martin, 2000; White & Mitchelmore, 1996). Other studies have found that solving non-geometric related problems is particularly challenging for students (e.g., Mkhatshwa, 2020a). Furthermore, findings from research indicate that the use of diagrams could be used to support students' quantitative reasoning when solving related rates problems (e.g., Engelke-Infante, 2021; Mkhatshwa, 2020a). I examined the availability (or lack thereof) of opportunities related to the aforementioned research findings in each textbook.

Illustrations of How Tasks were Coded Through the Lens of Quantitative Reasoning

In this section, I provide examples to illustrate how the tasks (examples and practice problems) were analyzed through the lens of quantitative reasoning.

Practice Problem 2 [Mathematics context] [Non-geometric] (Rogawski et al., 2019, p. 202):

$$\text{If } \frac{dx}{dt} = 2 \text{ and } y = x^3\text{, what is } \frac{dy}{dt} \text{ when } x = -4, 2, 6?$$

Practice Problem 2 is representative of practice problems I categorized as having a mathematics context, a related rates problem that does not provide quantitative reasoning opportunities such as interpreting quantities, and a non-geometric related rates problem because the equation relating the variables x and y (i.e., $y = x^3$) is not based on a geometric relationship such as the Pythagorean Theorem or the volume of a shape.

Example 3 [Real-world context] [Geometric] (Hughes-Hallett et al., 2021, pp. 254-255)**:** A spherical snowball melts in such a way that the instant at which its radius is $20\ cm$, its radius is decreasing at $3\ cm/min$. At what rate is the volume of the ball of snow changing at that instant?

Example 3 is representative of tasks I categorized as having a real-world context, as a task that requires simple mathematizing as finding the equation that relates the quantities of volume (V)

and radius (r) only requires recalling the volume of a sphere (i.e., $V = \frac{4}{3}\pi r^3$), and as a geometric related rates problem because the equation relating the quantities is based on the volume of a shape (i.e., sphere). Additionally, I categorized Example 3 as a task that provides an opportunity for students to assign a numerical value to the quantity representing the rate at which the volume of the snowball is changing (i.e., decreasing) at the instant when the radius is $20\ cm$. The radius is decreasing at a rate of $3\ cm/min$, and as a task that provides an opportunity for students to determine the units of measure (i.e., cm^3/min) for the aforementioned quantity.

Practice Problem 33 [Real-world context] [Non-geometric] (Barnett et al., 2019, p. 227): Suppose that for a company manufacturing calculators, the cost, revenue, and profit equations are given by

$$C = 90{,}000 + 30x$$
$$R = 300x - \frac{x^2}{30}$$
$$P = R - C$$

where the production output in 1 week is x calculators. If production is increasing at a rate of 500 calculators per week when production output is 6,000 calculators, find the rate of increase (decrease) in: (a) Cost, (b) Revenue, and (c) Profit.

Practice Problem 33 is another example of tasks in the textbooks that I categorized as having a real-world context. I further categorized this task as a non-geometric related rates problem because the equations relating the quantities x, P, R, and C (i.e., $C = 90{,}000 + 30x$, $R = 300x - \frac{x^2}{30}$, and $P = R - C$) are not based on geometric structures. I also categorized Practice Problem 33 as a task that does not require mathematizing as the equations relating the quantities x, P, R, and C are provided. Moreover, I categorized Practice Problem 33 as a task that provides opportunities for students to engage in the process of quantification (i.e., assigning numerical values to the quantities representing the rates at which C, R, and P are changing if production is increasing at a rate of 500 calculators per week when production output is 6,000 calculators). Finally, I categorized this task as providing an opportunity for students to make sense of and to determine the units of measure (i.e., dollars/week) for the aforementioned rate quantities.

Illustrations of How Tasks were Coded Through the Lens of Covariational Reasoning

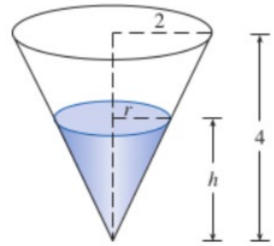
In this section, I provide examples to illustrate how the tasks were analyzed through the lens of covariational reasoning. I begin this section by noting that I categorized tasks that could not be analyzed through the lens of covariational reasoning (e.g., Practice Problem 2, reproduced in the preceding subsection) as tasks that do not provide opportunities to engage in covariational reasoning.

Example 3 [Real-world context] [Geometric] (Stewart et al., 2021, pp. 249-250): A water tank has the shape of an inverted cone with base radius $2\ m$ and height $4\ m$. If water is being pumped into the tank at a rate of $2\ m^3/min$, find the rate at which the water level is rising when the water is $3\ m$ deep.

Solution: We first sketch the cone and label it as in Figure 2. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

Figure 2

Accompanying Diagram-for Example 3



We are given that $\frac{dV}{dt} = 2\ m^3/min$ and we are asked to find $\frac{dh}{dt}$ when $h = 3\ m$. The quantities V and h are related by the equation $V = \frac{1}{3}\pi r^2 h$ but it is very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure 2 to write $\frac{r}{h} = \frac{2}{4}$, from which we get that $r = \frac{h}{2}$. The expression for V becomes $V = \frac{1}{3}\pi h \left(\frac{h}{2}\right)^2 = \frac{\pi}{12} h^3$. Now we can differentiate each side with respect to t: $\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$. Substituting $h = \ 3\ m$ and $\frac{dV}{dt} = \ 2m^3/min$, we have $\frac{dh}{dt} = \frac{4}{\pi(3^2)} * 2 = \frac{8}{9\pi}$. The water level is rising at a rate of $\frac{8}{9\pi} \approx$ $0.28\ m/min$.

Example 3 is representative of tasks I categorized as providing opportunities to engage at the coordination level of covariational reasoning (i.e., Level 1) because the quantities (V, r, and h) are changing simultaneously. The statement [in the solution of the task], "the water level is rising at a rate of $\frac{8}{9\pi} \approx 0.28\ m/min$", provides opportunities to engage at the direction and quantitative coordination levels of covariational reasoning (i.e., Levels 2 and 3). In particular, the remark about the water level rising as time elapses in the aforenoted statement constitutes engaging at the direction level of covariational reasoning. Quantifying the rate at which the water level is rising ($0.28\ m/min$) in the same statement constitutes engaging at the quantitative coordination level of covariational reason. I remark that none of the textbooks were coded at the highest two levels of covariational reasoning (i.e., average rate and instantaneous rate). Specifically, none of the expository sections or tasks included in the five textbooks analyzed in the present study provided opportunities for students to engage at the highest two levels of covariational reasoning.

Additionally, I categorized Example 3 as a task that has a real-world context, as a geometric related rates problem because the equation relating the quantities V, r, and h (i.e., $V = \frac{1}{3}\pi r^2 h$) is based on a geometric structure (i.e., a cone). This is also a task that provides an opportunity to engage in quantification (i.e., finding a numerical value of the rate at which the level of the water is rising), and as a task that requires simple mathematizing as formulating the equation relating the quantities V, r, and h does not require complex reasoning, as in, it can simply be recalled. Furthermore, I categorized Example 3 as a task that provides an opportunity for students to use a diagram (Figure 2) to support their quantitative reasoning when solving the related rates problem in the task. To clarify the coding process, I note that even though there are a few tasks (e.g., Example 3) I analyzed for both quantitative and covariational reasoning opportunities provided in the tasks, for the most part, these two codes (quantitative reasoning and covariational reasoning) are treated as mutually exclusive in the present study. I revisit this issue in the study limitations section of the manuscript.

Results

There are three primary results from this study. First, four of the five textbooks provide concise strategies (i.e., lists of three to seven steps) students could use when solving a related rates problem. The textbook by Hughes-Hallet et al. (2021) is the only textbook that does not provide a list of steps students could follow when solving a related rates problem. Second, all the textbooks provide plenty of opportunities to engage in quantitative reasoning via examples and practice problems on related rates problems. Third, there is a paucity of opportunities to engage in covariational reasoning through expository sections, examples, and practice problems on related rates problems, respectively, provided in all the textbooks. In addition, the few available opportunities are limited to low levels of covariational reasoning, namely coordination, direction, and sometimes quantitative coordination.

Definition of Related Rates Problems and Strategies for Solving these Problems

The definitions of a related rates problem given in the five textbooks are consistent with how a related rates problem is generally understood by the mathematics community in the United States, or how this type of problem is defined in the research literature (e.g., Mkhatshwa, 2020a). Specifically, according to one of the textbooks:

> In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the chain rule to differentiate both sides with respect to time (Stewart et al., 2021, p. 247).

Before giving the aforementioned definition of a related rates problem, Stewart and colleagues (2021) portrayed a picture of a related rates problem by giving the example of pumping air into a balloon. These authors remarked that in this example, it would be easier to measure directly the rate of increase of the volume of the balloon than the rate of increase of the radius of the balloon. These textbook authors went on to propose a seven-step problem-solving strategy (the most comprehensive problem-solving strategy compared to similar problem-solving strategies provided in three other textbooks) that can be used when solving a related rates problem. The following is a reproduction of this strategy (Stewart et al., 2021, p. 249):

> Step 1: Read the problem carefully.
> Step 2: Draw a diagram if possible.
> Step 3: Introduce notation. Assign symbols to all quantities that are functions of time.
> Step 4: Express the given information and the required rate in terms of derivatives.
> Step 5: Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
> Step 6: Use the chain rule [of differentiation] to differentiate both sides of the equation with respect to t [a time variable].
> Step 7: Substitute the given information into the resulting equation and solve for the unknown rate.

I note that the aforementioned steps are similar to steps previously used by students when tasked with solving geometric related rates problems (e.g., Engelke, 2007; Martin, 2000, Mkhatshwa, 2020a). I further note that usage of these steps is well illustrated through five examples in the textbook, one of which was reproduced in the Methods section. I remark that the regular calculus textbook by Rogawski et al. (2019) provides the least comprehensive problem-solving strategy (compared to three other textbooks) that could be used by students when solving a related rates problem. The following is a reproduction of this strategy (Rogawski et al., 2019):

> Step 1: Identify variables and the rates that are related.
> Step 2: Find an equation relating the variables and differentiate it.
> Step 3: Use given information to solve the problem.

Compared to Step 6 in Stewart et al.'s (2021) problem solving strategy, among other things, Step 2 in Rogawski et al.'s (2019) problem solving strategy does not specify the type of differentiation [chain rule] that is to be used after finding the equation that relates the quantities involved in the problem.

Opportunities to Engage in Quantitative Reasoning

Expository Sections. None of the nine expository sections noted in Table 2 provide opportunities to engage in quantitative reasoning. That is, the expository sections in the five textbooks do not provide opportunities to interpret physical quantities, to determine units of measure for physical quantities, or to engage in the process of quantification. I note, however, that all the expository sections came close to providing something we would consider to be opportunities to engage in quantitative reasoning. In the life sciences textbook, for example, Greenwell et al. (2015) posed the following rhetorical question to highlight the importance of related rates problems in the life sciences: When a skier's blood vessels contract because of the cold, how fast is the velocity of the blood changing? These textbook authors went on to make the following remark prior to providing examples on related rates problems:

> It is common for variables to be functions of time; for example, sales of an item may depend on the season of the year, or a population of animals may be increasing at a certain rate several months after being introduced into an area. Time is often present implicitly in a mathematical model, meaning that derivatives with respect to time must be found by the method of implicit differentiation discussed in the previous section (p. 343).

While none of the quantities (e.g., sales of an item, population of animals, the rate of change of the population of animals) needed to be interpreted in the preceding pair of statements, to emphasize the importance of units when making sense of quantities and relationships between quantities, one can argue that Greenwell et al. (2015) could have used, for example, antelopes per year as a unit of measure for the quantity that represents the rate at which the population of animals [e.g., antelopes] is increasing. Similar remarks were made by the authors of the other textbooks considered in this study.

Examples. As can be seen in Table 3, all the examples presented in the related rates section of each of the five textbooks provide ample opportunities to engage in quantitative reasoning (i.e., these examples have real-world contexts). Specifically, the examples generally provide opportunities to interpret physical quantities, to assign units of measure to these quantities, or to engage in the process of quantification. Example 1 is a typical example (in addition to the two Examples 3s that were reproduced in the Methods section) that provides opportunities to engage in the process of quantification, and to make sense of quantities, relationships among quantities, and units of measure for quantities, respectively:

Table 3

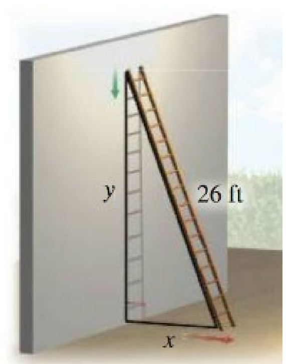
Classification of Examples by Type of Context

Textbook Name	Count of examples with a real-world context	Count of examples with a mathematics context
Calculus: Early Transcendentals (9th ed)	5	0
Calculus: Early Transcendentals (4th ed)	5	0
Single Variable Calculus (8th ed)	4	0
Calculus for the Life Sciences (2nd ed)	6	0
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	4	0
Total	**24**	**0**

Example 1 [Real-world context] [Geometric] (Barnett et al., 2019, p. 222): A 26-foot ladder is placed against a wall as shown in Figure 3.

Figure 3

Diagram that Accompanies Example 1



If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?

This example provides an opportunity to make sense of how several quantities (the vertical distance of the ladder denoted by the variable y in Figure 3, the horizontal distance of the ladder denoted by the variable x in Figure 3, and the rates of change of x and y as the top of the ladder is sliding down the wall) are related. Furthermore, the example provides an opportunity to determine units of measure for the unknown quantity (i.e., the rate at which the bottom of the ladder is moving away from the wall at the instant when the bottom of the ladder is 10 feet away from the wall). It also provides an opportunity to engage in the process of quantification (i.e., determine a numerical value for the quantity that represents the rate at which the bottom of the ladder is moving away from the wall at the instant when the bottom of the ladder is 10 feet away from the wall). Lastly, I interpreted the inclusion of Figure 3 in Example 1 as a means of supporting students' reasoning about relationships among the quantities involved in the example. As can be seen in Table 4, most of the examples in the five textbooks have accompanying diagrams to support students' quantitative reasoning when working through these examples.

Table 4

Count of Examples With or Without Accompanying Diagrams

Textbook Name	Count of examples with accompanying diagrams	Count of examples without accompanying diagrams
Calculus: Early Transcendentals (9th ed)	4	1
Calculus: Early Transcendentals (4th ed)	5	0
Single Variable Calculus (8th ed)	3	1
Calculus for the Life Sciences (2nd ed)	4	2
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	2	2
Total	**18**	**6**

Even though the examples on related rates problems given in the five textbooks are rich in terms of opportunities to engage in quantitative reasoning, as can be seen in Table 5, a majority of the examples are geometric related rates problems.

Table 5

Count of Geometric Versus Non-Geometric Related Rates Examples

Textbook Name	Count of geometric examples	Count of non-geometric examples
Calculus: Early Transcendentals (9th ed)	5	0
Calculus: Early Transcendentals (4th ed)	5	0
Single Variable Calculus (8th ed)	3	1
Calculus for the Life Sciences (2nd ed)	4	2
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	3	1
Total	**20**	**4**

Consequently, mathematizing these examples is straightforward, as it typically involves recalling formulas that relate the quantities involved in these tasks. In Example 1, the equation relating the length of the ladder (26 ft), the quantity representing the vertical distance of the ladder (y), and the quantity representing the horizontal distance of the ladder (x) is given by the Pythagorean Theorem (i.e., $x^2 + y^2 = 26^2$).

Practice Problems. As can be seen in Table 6, a great majority of the practice problems in the five textbooks provide numerous opportunities to engage in quantitative reasoning (i.e., they have real-world contexts). The following is a reproduction of a geometric related rates problem, typical of the five textbooks, that provides opportunities to engage in quantitative reasoning:

Table 6

Classification of Practice Problems by Type of Context

Textbook Name	Count of practice problems with a real-world context	Count of practice problems with a mathematics context
Calculus: Early Transcendentals (9th ed)	51	2
Calculus: Early Transcendentals (4th ed)	43	2
Single Variable Calculus (8th ed)	64	5
Calculus for the Life Sciences (2nd ed)	26	10
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	42	6
Total	**226**	**25**

Practice Problem 13 [Real-world context] [Geometric] (Rogawski et al., 2019, p. 203): At a given moment, a plane passes directly above a radar station at an altitude of $6\ km$.

(a) The plane's speed is $800\ km/h$. How fast is the distance between the plane and the station changing half a minute later?
(b) How fast is the distance between the plane and the station changing when the plane passes directly above the station?

Parts (a) and (b) prompt students to engage in quantification (i.e., to quantify the rate at which the distance between the plane and the station is changing). Students are also expected to determine the units of measure for the specified quantities in parts (a) and (b). Mathematizing this problem (and many other geometric related rates problems found in the five textbooks) is not challenging as it involves using the Pythagorean Theorem. Practice Problem 18, in addition to Practice Problem 33 reproduced in the Methods section, is an example of the few non-geometric related rates problems found in the five textbooks.

Practice Problem 18 [Real-world context] [Non-geometric] (Greenwell et al., 2015, p. 348): The energy cost of horizontal locomotion as a function of the body weight of a marsupial is given by $E = 22.8w^{-0.34}$, where w is the weight (in kg) and E is the energy expenditure (in $kcal/kg/km$). Suppose that the weight of a $10\ kg$ marsupial is increasing at a rate of $0.1kg/day$. Find the rate at which the energy expenditure is changing with respect to time.

Among other things, Practice Problem 18 provides an opportunity to quantify the unknown quantity (i.e., the rate at which the energy expenditure is changing with respect to time. Furthermore, this practice problem provides an opportunity to make sense of the units of measure for the aforementioned unknown quantity. As with all the other non-geometric related rates problems provided in the five textbooks, students do not have to mathematize this task as the equation [$E = 22.8w^{-0.34}$] relating the quantities E and w is given as part of the statement of the problem. In general, I found no trend in the frequency or amount of available opportunities to work with non-geometric related rates problems presented in the five textbooks. Table 7 displays this information.

Table 7

Count of Geometric Versus Non-Geometric Related Rates Practice Problems

Textbook Name	**Count of geometric practice problems**	**Count of non-geometric practice problems**
Calculus: Early Transcendentals (9th ed)	47	6
Calculus: Early Transcendentals (4th ed)	38	7
Single Variable Calculus (8th ed)	35	34
Calculus for the Life Sciences (2nd ed)	12	24
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	25	23
Total	**157**	**94**

Specifically, while the number of non-geometric related rates problems is extraordinarily low in the Stewart et al. (2021) and Rogawski et al. (2019) textbooks, the proportion of geometric and non-geometric related rates problems in the Hughes-Hallett et al. (2021) and Barnett et al. (2019) textbooks is nearly the same. Furthermore, a majority of the problems in the Greenwell et al. (2015) textbook are non-geometric related rates problems. Additionally, opportunities promoting the use of diagrams to make sense of quantities and relationships among quantities while working with related rates problems are extremely low in all five textbooks analyzed in this study. Table 8 displays this information.

Table 8

Count of Practice Problems With or Without Accompanying Diagrams

Textbook Name	**Count of practice problems with accompanying diagrams**	**Count of practice problems without accompanying diagrams**
Calculus: Early Transcendentals (9th ed)	10	43
Calculus: Early Transcendentals (4th ed)	14	31
Single Variable Calculus (8th ed)	9	60
Calculus for the Life Sciences (2nd ed)	6	30
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	2	46
Total	**41**	**210**

Opportunities to Engage in Covariational Reasoning

Expository Sections. Opportunities to engage in covariational reasoning in the nine expository sections (identified in Table 2) on related rates problems in the five textbooks are limited to the lowest two levels of covariational reasoning, namely Level 1 (coordination) and Level 2 (direction). The following is a reproduction, from one of the textbooks, of an exemplary opportunity to engage in covariational reasoning in the expository section of a textbook:

> Union workers are concerned that the rate at which wages are increasing is lagging behind the rate of increase in the company's profits. An automobile dealer wants to predict how much an anticipated increase in interest rates will decrease his rate of sales. An investor is studying the connection between the rate of increase in the Dow Jones average and the rate of increase in the gross domestic product over the past 50 years. In each of these situations, there are two quantities-wages and profits, for example-that are changing with respect to time. We would like to discover the precise relationship between the rates of increase (or decrease) of the two quantities. We begin our discussion of such related rates by considering familiar situations in which the two quantities are distances and the two rates are velocities (Barnett et al., 2019, p. 222).

This remark provides an opportunity to engage in Level 1 of covariational reasoning as it creates an awareness of two quantities (wages and profits) changing in tandem. It also provides an opportunity to engage in Level 2 of covariational reasoning as it speaks of the direction of change (increasing or decreasing) of the aforementioned quantities.

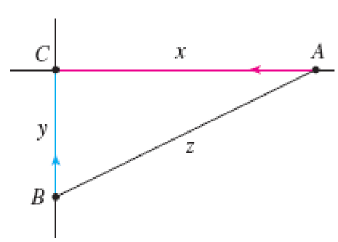
Examples. All the related rates examples from the five textbooks identified in Table 2 provide opportunities to engage in covariational reasoning. However, these opportunities are limited to the lowest levels of covariational reasoning, namely Level 1 (coordination), Level 2 (direction), and Level 3 (quantitative coordination). The following is a reproduction of a typical example from one of the textbooks, in addition to Example 3 that was reproduced in the Methods section, and a discussion of the opportunities to engage in covariational reasoning provided in this example:

Example 4 [Real-world context] [Geometric] (Stewart et al., 2021, p. 250): Car A is traveling west at $50\ mi/h$ and car B is traveling north at $60\ mi/h$. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is $0.3\ mi$ and car B is $0.4\ mi$ from the intersection?

Solution: We draw Figure 4, where C is the intersection of the roads.

Figure 4

Accompanying Diagram-for Example 4



At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles. [It should be noted that although Figure 4 is as a generic diagram that represents the given real-world scenario, in the specific problem given in Example 4, the horizontal distance between A and C is 0.3 miles, and the vertical distance between B and C is 0.4 miles].

We are given that $\frac{dx}{dt} = -50\ mi/h$ and $\frac{dy}{dt} = -60\ mi/h$. (The derivatives are negative because x and y are decreasing). We are asked to find $\frac{dz}{dt}$. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$x^2 + y^2 = z^2$$

Differentiating each side with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dt}{dt}$$
$$<=> \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \qquad z \neq 0$$

When $x = 0.3\ mi$ and $y = 0.4\ mi$, the Pythagorean Theorem gives $z = 0.5\ mi$, so

$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)]$$
$$= -78\ mi/h$$

The cars are approaching each other at a rate of 78 mi/h.

In the preceding solution to Example 4, the statement "at a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars" provides an opportunity to engage in Level 1 (coordination) of covariational reasoning as it provides evidence of the three quantities x, y, and z changing simultaneously with changes in time. In the same solution, the remark that "the derivatives are negative because x and y are decreasing" provides an opportunity to engage in Level 2 (direction) of covariational reasoning. Finally, quantifying the quantities dx/dt and dy/dt through the comment "we are given that $\frac{dx}{dt} = -50\ mi/h$ and $\frac{dy}{dt} = -60\ mi/h$" provides an opportunity to engage in Level 3 (quantitative coordination) of covariational reasoning.

Practice Problems. Practice problems on related rates problems in the five textbooks either do not provide opportunities to engage in covariational reasoning at all such as Practice Problem 2 that was reproduced in in the Methods section, or they provide opportunities to engage at the coordination and direction levels of covariational reasoning (i.e., Levels 1 and 2 of covariational reasoning). Indeed, of the 251 practice problems (see Table 2), 25 practice problems do not provide opportunities to engage in covariational reasoning and the remaining 226 problems provide opportunities to engage at the coordination and direction levels of quantitative reasoning. The following is a reproduction of a representative practice problem from the five textbooks:

Practice Problem 39 [Real-world context] [Geometric] (Hughes-Hallett et al., 2021, p. 260): The radius of a spherical balloon is increasing by $2\ cm/sec$. At what rate is air being blown into the balloon at the moment when the radius is $10\ cm$? Give units in your answer.

This problem provides an opportunity to engage in Level 1 (coordination) of covariational reasoning in that it presents the opportunity to visualize how the quantities of radius and volume are changing in tandem with changes in time. In addition, the question clearly states that the radius is increasing [and while not stated, it can be inferred that the volume is increasing as air is blown into the balloon], thus providing an opportunity to engage in Level 2 (direction) of covariational reasoning. Like Practice Problem 39, most of the practice problems in the five textbooks are focused on calculational knowledge rather covariational reasoning. That is, they tend to emphasize performing calculations over posing questions that promote making sense of how different quantities are changing in relation to each other as time changes.

Discussion and Conclusions

Even though the five textbooks do not have opportunities to engage in quantitative reasoning in their expository sections on related rates problems, the textbooks provide ample opportunities to engage in quantitative reasoning through 24 examples and 226 (out of 251) practice problems on related rates problems, respectively. The prevalence of opportunities to engage in quantitative reasoning in the textbooks is in compliance with growing calls from several researchers and mathematics educators to include such opportunities in undergraduate mathematics education (e.g., Castillo-Garsow, 2012; Moore, 2014; Thompson, 2011). Arguably, the fact that opportunities to engage in quantitative reasoning are plentiful may suggest that some of the previously reported students' difficulties, such as interpreting quantities (e.g., Azzam et al., 2019; Mkhatshwa, 2020a, Kottath, 2021) and making sense of relationships among quantities (e.g., White & Mitchelmore, 1996), with engaging in quantitative reasoning when solving related rates problems may originate from other sources (e.g., classroom instruction), and not necessarily from calculus textbooks.

Findings from previous research on related rates problems indicate that diagrams [pictures of situations] are helpful when solving geometric related rates problems (e.g., Engelke-Infante, 2021; Mkhatshwa, 2020a). In general, it is commendable that all the textbooks considered in this study provide a substantial number of opportunities (via examples) to work with diagrams when solving related rates problems. Specifically, 18 of the 24 examples on related rates problems presented in the five textbooks have accompanying diagrams, thus promoting the use of diagrams when working with related rates problems. On the contrary, opportunities promoting the use of diagrams via practice problems when solving related problems are disproportionately low in all five textbooks. In particular, of the 251 practice problems on related problems found in the five textbooks, only 41 practice problems have accompanying diagrams. I thus recommend that textbook selection committees in mathematics departments consider, among other things, the proportion of examples and practice problems providing opportunities to work with diagrams when adopting calculus textbooks. Similarly, calculus instructors are encouraged to regularly use diagrams (when appropriate) in their teaching of related rates problems in calculus. It would also benefit students if instructors could include explicit prompts on homework assignments or even exams (on related rates), encouraging students to create and use diagrams (when appropriate) to support their quantitative reasoning when solving related rates problems.

Mathematizing a great majority of the geometric-related rates tasks found in the five textbooks is, for the most part, straightforward and often involves using slight variations of the Pythagorean theorem or recalling geometric formulas such as the formula for the volume of a sphere. In addition, nearly all the non-geometric related rates problems found in the five textbooks do not need to be mathematized, as the equations relating the quantities involved in these problems are provided. Of the five textbooks considered in this study, the proportion of non-geometric related rates problems (compared to geometric related rates problems) was extraordinarily low in two of the textbooks, about the same in two other textbooks, and significantly high in one other textbook. A common theme from a growing number of studies on related rates problems is that mathematizing these types of problems is often a challenge for many students in calculus (e.g., Azzam et al., 2019; Jeppson, 2019; Martin, 2000; Mkhatshwa, 2020a; White & Mitchelmore, 1996). To this end, I recommend that calculus textbook authors consider including a fair balance of geometric and non-geometric related rates problems in their textbooks, and most importantly, including related rates problems that require engaging in deep and meaningful aspects of quantitative reasoning that go beyond simply recalling and using geometric formulas when mathematizing these problems. The same consideration applies to calculus instructors in their teaching of related rates problems, or textbook selection committees in mathematics departments, when adopting calculus textbooks for their departments.

Opportunities to engage in covariational reasoning (Carlson et al., 2002) provided in the five textbooks are not only minimal, but also limited to the lowest levels of covariational reasoning, namely coordination, direction, and quantitative coordination. Specifically, I did not find any opportunities to engage in the highest (i.e., more sophisticated) levels of covariational reasoning, namely average rate and instantaneous rate in the expository sections, examples, and practice problems on related rates problems, respectively, included in the five textbooks. Findings from research indicate that students' covariational reasoning abilities are typically limited to the lowest levels of covariational reasoning when solving related rates problems (e.g., Engelke, 2007). Other research has found that students show little or no evidence at all of engaging in covariational reasoning when dealing with derivatives, which are crucial elements of related rates problems (e.g., Carlson et al, 2002; Jones, 2017; Nagle et al., 2013).

In light of the fact that opportunities to engage in covariational reasoning, let alone opportunities to engage in the highest levels of covariational reasoning, are scanty in the five textbooks, I recommend that calculus textbook authors include plenty of opportunities to engage in covariational reasoning when creating expository sections, examples, and practice problems on related rates problems. This is especially important because evidence from research indicates that most student learning is often directed by the textbook rather than the instructor (e.g., Alajmi, 2012; Begle, 1973; Kolovou et al., 2009, Törnroos, 2005; Wijaya et al., 2015). In fact, Reys et al. (2004) posited that the presentation of instructional content during course lectures closely follows the presentation of such content in mathematics textbooks, an argument supported by other scholars (e.g., Blazar et al., 2020; Polikoff et al., 2021). Furthermore, I recommend that textbook selection committees adopt textbooks that provide such opportunities in abundance in light of the crucial role that covariational reasoning plays in students' understanding of calculus topics, including related rates problems. Finally, I recommend that calculus instructors create and use, during classroom instruction, more tasks that could support students in developing strong covariational abilities (i.e., support them in engaging in the highest levels of covariational reasoning). This could include designing tasks that require students not only to create diagrams, but also to make sense of these diagrams to successfully solve related rates problems. Additionally, this might mean calculus instructors will have to design and use related rates problems that have realistic and essential contexts during classroom instruction. This is particularly important because evidence from a recent study on the teaching of related rates problems indicated that related rates problems were not varied and tended to be similar from one calculus textbook to another (Mkhatshwa, 2023).

In conclusion, I note that the five calculus textbooks examined in this study are arguably representative of a great majority of widely used textbooks in the teaching of regular, business, and life sciences calculus, respectively, in the United States. I further note that results from the present study are, to a great extent, consistent with findings from my recent study (Mkhatshwa, 2022) that examined learning opportunities about ordinary and partial derivatives provided by two calculus textbooks. Specifically, both studies have found that calculus textbooks by and large provide enough quantitative reasoning opportunities, and that there is a deficiency of covariational reasoning opportunities, especially opportunities to engage in the highest levels of covariational reasoning, in the same textbooks. Based on the findings of these two studies and a growing number of calls from renowned scholars and educators to include covariational reasoning opportunities in the study of calculus, I appeal to calculus textbook authors to substantially increase covariational reasoning opportunities in their textbooks in virtually every topic (e.g., derivatives, related rates problems, differentials, optimization problems, etc.). This is especially true for opportunities to engage in the highest levels of covariational reasoning, namely average rate and instantaneous rate, which are currently lacking in most widely used calculus textbooks.

Study Limitations

I conclude this paper by highlighting the study limitations. First, the textbooks analyzed in the present study are widely used in the teaching of calculus in the United States. Consequently, findings of the present study may not extend beyond the United States. It might be important for future research to examine similar opportunities to learn provided by other widely used calculus textbooks in other parts of the world. Second, Pershing (2002) remarked that document analysis can be used as a stand-alone data-collection procedure or as a precursor to collecting new data. In this study, document analysis was used as a stand-alone data collection procedure. I, however, posit that there might be added value in using document analysis in conjunction with other methodologies. For instance, it might be helpful to present researchers' findings from conducting a document analysis alongside perspectives of the authors of the documents that were analyzed. In the context of the present study, it would have been beneficial to present the textbooks author's perspectives (obtained via interviews or questionnaires) alongside the results obtained by analyzing the five textbooks examined in the study. Third, the quantitative reasoning and covariational reasoning codes used in the present study are mostly mutually exclusive. In other words, I did not consider tasks that provide opportunities to engage in both quantitative and covariational reasoning in greater detail. This could be a subject for future research. This is especially compelling because these two codes may not necessarily be mutually exclusive. Specifically, geometric related rates problems that are situated in real-world contexts provide opportunities for engaging in the two modes of reasoning, namely quantitative and covariational. Fourth, the data was coded by one researcher. Consequently, inter-rater reliability was not established. Follow-up research will likely involve several researchers to establish inter-rater reliability, among other potential benefits of conducting collaborative research.

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