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CONTENTS

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International Collaboration in Science and Mathematics Education

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The *Electronic Journal for Research in Science & Mathematics Education* is the flagship journal of the International Consortium for Research in Science & Mathematics Education (ICRSME). ICRSME was conceived by Dr. Arthur L. White in 1983 as a result of working on various projects in Central America and the Caribbean under the auspices of The Ohio State University and the United States Information Agency (USIA). The USIA was in effect from 1953 through 1999 (National Archives, 2020). The two-pronged mission of the USIA guided its activities:

> To seek to inform others about American life and values, policies, and interests as a nation; and, if possible, to eliminate misperception and move others to action in ways that serve the national interest; and second that mutual understanding borne of people-to-people communication matters, and that USIA should serve as a facilitator to bring Americans and their academic and other nongovernmental sector institutions into substantive contact with influential counterparts abroad through exchanges and other programs. (United States, p. 718)

By 1985, a variety of cooperative and collaborative projects were underway across institutions and countries, leading to ICRSME's first consultation in 1986 hosted in Port of Spain, Trinidad and Tobago. Subsequently, Dr. White and Dr. Donna F. Berlin organized 14 consultations across Central and South America and the Caribbean over three decades.

The mission of ICRSME is the advancement of science and mathematics education in the participating countries. This mission is based on the premise that all peoples can benefit from the knowledge and experiences of their local, national, and international colleagues. To serve the mission, the consortium model includes five interrelated goals:

1. Designing, facilitating, and conducting research and development toward the improvement of science and mathematics teaching and learning
2. Developing academic exchange programs between universities in order to broaden the educational experiences of students and faculty
3. Acting as an impetus in establishing ties between the local, state, and national educational associations in the participating countries
4. Identifying the particular science and mathematics education needs and issues facing current and emerging under-represented populations in the participating countries and directing research and development to address those needs and issues
5. Promoting collaborative efforts among scholars in the participating countries

As ICRSME continues to evolve, the organization plans to consider its mission and these goals and how they can be met. This editorial serves as an introduction to a series of editorials about fostering effective and genuine international collaboration in science and mathematics education.

Genuine Collaboration

In the process of developing genuine and productive collaboration, partners may encounter challenges. Such issues include differences in cultures and norms of diverse settings, variation in points of view and body-of-knowledge of persons involved, power structures, disparate motivations for involvement, weak communication about the goals of the work, and a lack of trust between partners (Adamson & Walker, 2011; Barnett et al., 2006; Buysse et al., 2003; Sim, 2010). One way to counter these challenges is to be purposeful in identifying the type of collaboration desired and the approach to building this collaboration, ensuring critical components are present at the outset.

The construct of *communities of practice* can serve as a framework to guide collaborative work. Communities of practice are "groups of people who share a concern, set of problems, or passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (Wenger et al., 2002, p. 4). Wenger et al. (2002) describe three core characteristics of communities of practice: "a domain of knowledge, which defines a set of issues; a community of people who care about this domain; and the shared practice that they are developing to be effective in their domain" (p. 27). For instance, ICRSME is a community of people who care about the learning and teaching of science and mathematics. Moving forward, the organization plans to consider how to purposefully develop a shared practice.

Two critical premises ground the formation of communities of practice: a shared goal (Clausen et al., 2009) and the co-construction of knowledge (Palinscar et al., 1998; Sim, 2010). The establishment of both of these conditions has been found to result in genuine collaboration with opportunities to learn for all parties involved (Quebec Fuentes & Spice, 2017). Quebec Fuentes and Spice differentiate between *shared but given goals* and *shared beyond given goals*. For example, with ICRSME, a shared but given goal is the advancement of science and mathematics education. A shared beyond given goal would focus on an aspect of the learning and teaching of science and mathematics in a particular setting and relevant to the parties involved. In other words, a shared beyond given goal is a mutually established endeavor centered on a specific area of need (Buysee et al., 2001).

Effective collaborations also move away from an authoritarian, hierarchical, or colonialist model of knowledge dissemination. A foundational aspect of communities of practice is a shift from attention to individual ideas to group interactions (Buysee et al. 2003). In other words, "learning is viewed as distributed among many participants within the community in which people with diverse expertise (i.e., experts, novices, and those in between) are transformed through their own actions and those of other participants" (Buysee et al., 2003, p. 266). This concept of distributed expertise (Pugach, 1999) emphasizes the co-construction of knowledge with various stakeholders sharing ideas and perspectives (Palinscar et al., 1998; Sim, 2010).

Factors that enable the development of a shared goal and an environment for the co-construction of knowledge include conflict, communication, trust, and reflection. Conflict can be viewed as an undesirable *situation* or as an *ongoing process* that has the potential to lead to learning (Achinstein, 2002, p. 425). If conflict is perceived as a problem, participants avoid examining their beliefs and assumptions and instead establish a *culture of nice* (MacDonald, 2011, p. 46) through *feigned politeness* (Hargreaves, 2001), *superficial effort* (Barnett et al., 2006), and *contrived collegiality* (Hargreaves, 1994). On the other hand, if conflict is embraced, participants "acknowledge, solicit, and own conflict by critically reflecting upon differences of belief and practice," opening up space for "active dissent and opportunities for alternative views," the transformation of the status quo, and organizational learning (Achinstein, 2002, p. 441).

To support growth through the process of conflict, norms of communication must be established. The particular means of communication within a community are unique (Sim, 2010) since the community builds a shared language through their open and critical dialogue (Wenger et al., 2002). Additionally, trust amongst participants is essential for communication (Barnett et al., 2006; Palinscar et al., 1998). "Trust can be established if the community assumes that responsibility for understanding is shared, and authority for knowing is internal and collective" (Palinscar et al., 1998, p. 9). Trust and respect within a community allows for critical reflection centered on the collaborative sharing, listening, challenging, and reconstructing of ideas (Wenger et al., 2002). Reflection can be integrated into the discourse through informal or purposefully structured processes (Adamson & Walker, 2011; Quebec Fuentes & Spice, 2017). The purpose of the reflection is twofold; members of the community consider their progress toward their joint learning endeavor as well as monitor the collaborative process itself (Buysee et al., 2001; Quebec Fuentes & Spice, 2017).

International Collaboration: Challenges and Opportunities

International collaboration faces the same aforementioned challenges as well as some additional considerations and obstacles. Some issues are logistical, such as working around time differences and different academic calendars (Peled & Rozansky, 2014). Other deeper considerations address who is included and how they are involved. The concept of *border politics* is the process of "negotiating the bounds of membership and beliefs of a given community" (Achinstein, 2002, p. 426), and Atweh and Keitel (2007) examine border politics from a social justice lens.

In particular, Atweh and Keitel (2007) examine five signs of social injustice in international collaboration (exploitation, marginalization, powerlessness, cultural imperialism, and violence). Exploitation in research endeavors occurs when the accomplishments and perspectives of one group is furthered to the detriment of others. Additionally, research foci and methods of some countries are valued more in the international community, pushing the problems of practice (and ways of addressing them) in other countries to the margins. Further marginalization stems from language and economics. For instance, the primary language(s) used to communicate within a community could force members to the periphery of or completely exclude them from involvement (Adamson & Walker, 2011). Some academics may not be able to participate in international scholarly activities, such as conferences, due to their cost. This lack of involvement results in powerlessness. Cultural imperialism is then evidenced in "the non-critical transfer of curricula and research results from one country with a certain perceived higher status to another" (Atweh & Keitel, 2007, p. 14). Lastly, linking economic support from more affluent countries to cultural imperialism is viewed as symbolic violence.

International collaboration has the potential to counter these injustices if grounded in the premise that people learn from each other through such collaboration (Atweh & Keitel, 2007). First, the borders of the community need to be expanded to include members from different cultures and contexts. Communication and conflict allows for the negotiation, rather than the imposition, of the border politics by members of the community (Achinstein, 2002). Second, policies and practices at all levels (local, state/provincial, national, and international) are influenced by historical, political, economic, and social circumstances of a setting. When international peers share their funds of knowledge and compare these influences across locations through dialogue, a greater understanding of each context develops (e.g., Winton & Pollack, 2014).

Lastly, international groups must regularly reflect on their collaboration to ensure that it maintains socially just actions.

> International contacts and exchanges in mathematics and mathematics education have … increased in the new age of globalization and will continue to exponentially increase in the future with further developments in technology, ease of travel and population movements. While we do not construct such contacts as necessarily either good or bad, the outcomes of these processes should be carefully scrutinized world wide as to the benefits and losses that

> might arise from them. This can only be achieved through deliberate and targeted reflection and debate. (Atweh et al., 2003, p. 224).

The following questions can guide such deliberation (Atweh & Keitel, 2007):

1. Who is included in the international collaboration?
2. How are the various members' included in the activity?
3. Are decisions being made in a just and fair way?
4. Are the means to work together effectively and with equal rights collaboratively considered?
5. Who benefits from the international collaboration?
6. Whose views are expressed in the products of the international collaboration?
7. Whose knowledge is being represented in the international collaboration?

As indicated by the fourth question, such inquiries should be consistently interrogated by all partners.

Conclusion

ICRSME and EJRSME are committed to promoting genuine international collaboration to advance science and mathematics education. This EJRSME editorial begins a series that will examine international collaborations and reflect on the opportunities and potential pitfalls that such relationships can present. As the flagship journal of ICRSME, we hope these editorials will inspire you to consider ways in which you can engage with colleagues in genuine collaboration through future international consultations and virtual conferences.

The theme of our upcoming virtual conference, taking place on March 12, 2022, is *International Collaboration in Science and Mathematics Education*. We will feature Dr. Grace Bascope, from the Botanical Research Institute of Texas (BRIT), sharing *Lessons Learned from Collaborative Place-Based Learning Programs in Yucatan, Mexico and Belize* as well as Dr. Ricardo Lleonart, del Instituto de Investigaciones Científicas y Servicios de Alta Tecnología de Panamá, sharing *INDICASAT AIP - A Model Institute for Innovation in Research and Education*. We look forward to learning from these international collaborators as well as the many ICRSME friends who will be presenting both virtually (asynchronously) and in round table discussion rooms.

At the virtual conference, we will be announcing a newly formed ICRSME ad hoc committee that will be focused on answering some of the questions posed in this editorial. This diverse and international committee will examine the ICRSME mission statement and goals as well as other ICRSME activities and opportunities to determine if we are doing all we can to foster the *genuine collaboration* between participating countries that we desire.

As we strive to deepen our collective understanding of genuine international collaboration in science and mathematics education, we hope to learn from those who have engaged in such work already. We encourage you to share your experiences through multiple venues hosted by ICRSME (*ICRSME Newsletter*, EJRSME, virtual conferences, and biennial consultations).

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Mathematical Representations in the Teaching and Learning of Geometry: A Review of the Literature from the United States

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ABSTRACT

This paper presents a synthesis of the literature exploring the teaching and learning of geometry and the role that mathematical representations can play in enriching geometry experiences for our students. Geometry is the only content domain to be taught in all PK-12 grades, however, from historical trends in international assessment data, it continues to be a low scoring area for students in the United States. This paper is organized by the following: (1) theories guiding the teaching and learning of geometry in the U.S.; (2) teaching and learning of geometry in the U.S.; and (3) the role of mathematical representations in geometry. In order for students to appreciate and experience the wonder, joy, and beauty of geometry in a consistent and coherent manner, they need geometry learning experiences that leverage high quality tasks with opportunities in translating *between* and *within* multiple representations, and engage them in discovering connections within geometry, between geometry and the other mathematics content domains, and between geometry and their world.

Keywords: geometry, high quality tasks, mathematical representations, mathematics education

Introduction

In the 2001 National Council of Teachers of Mathematics (NCTM) Yearbook, Albert Cuoco challenged the mathematics education community to think beyond ideas of content and pedagogy, to how our students learn (NCTM, 2001). Most recently, one way NCTM has addressed this call is through the introduction of the eight mathematics teaching practices (NCTM, 2014). The aforementioned practices provide students the opportunity to access mathematics through multiple entry points while leveraging multiple mathematical representations (i.e. visual, symbolic, verbal, contextual, and physical) (Lesh et al., 1987). During the past decades, mathematical representations have been defined in multiple ways (Goldin, 2014; Huinker, 2015; NCTM, 2014). For the purpose of this paper, mathematical representations will refer to the five types that were initially defined by Lesh and colleagues (1987), which we describe in more depth later in this paper. Huinker and Bill (2017) refer to the importance of students using multiple mathematical representations - both *between* representation types and *within* the same representation type. The ways in which these connections among mathematical representations can be leveraged - specifically in geometry instruction - will also be discussed in this paper.

Some effective uses of mathematical representations include connecting instruction with students' experiences and interests (NCTM, 2018). Teaching geometry is crucial in facilitating student opportunities to make connections with the real world (Usiskin, 1980), in addition to experiencing geometry in an integrated and active manner capitalizing on the wonder, joy and beauty of examining the world (NCTM, 2020a). The study of geometry and measurement provides rich opportunities for children to both explore and visualize the two- and three-dimensional, and represent objects and the relationships between them, and enrich and connect geometrical ideas to other mathematical domains and the world around them (NCTM 2020a, 2020b).

Gonzáles and Herbst (2006) identify four aims for the teaching of geometry: (a) a *formal* argument: geometry teaches to use logical reasoning; (b) a *utilitarian* argument: geometry serves to prepare students for the workplace; (c) a *mathematical* argument: geometry for the experience and the ideas of mathematicians; and (d) an *intuitive* argument: geometric expression helps students interpret their experiences in the world. Prior research (i.e. International Commission on Mathematical Instruction, ICMI, 1995) has shown that there is no linear, hierarchical path from beginning to more advanced geometry – geometric ideas must be examined, reconsidered, reimagined, and refined at different stages from different viewpoints.

Among mathematicians and mathematics educators, there is widespread agreement that teaching geometry should start at an early age and should continue throughout the entire mathematics curriculum (ICMI, 1995). This is also illustrated in the Common Core State Standards for Mathematics (2010) progression (and other similar college and career readiness standards) which list geometry as the only domain taught in all PK-12 grades. This makes it evident that current reform-based curriculum supports revisiting geometric ideas, however, historically speaking, "Geometry has been treated solely as geometry and not as a subject, which in addition to being a splendid example of deductive reasoning, important and interesting in itself, can also serve the purpose of creating a critical attitude of mind toward deduction and thinking in general" (NCTM, 1940, p. 39). In many countries, geometry has also lost its former central position in mathematics teaching – the subject is often somewhat ignored or confined to the teaching of facts about figures and their properties (ICMI, 1995). In the U.S., students revisit the subject every year, yet, they are often given too little exposure to geometrical thinking in grades K–8, particularly in the middle grades, so their understanding of geometry does not always develop to deeper levels of analysis (Clements, 2003; Clements & Battista, 1992; Driscoll, 2007; Steele, 2013). Several researchers have supported the idea that an increased focus on researching and understanding the place of geometry in curriculum would be well advised (Fuys et al., 1988; Sinclair & Bruce, 2015).

While ideas about the use of mathematical representations have been researched, as well as about geometry curriculum, there is little literature that synthesizes both. Individual representations cannot fully describe a mathematical construct, and each has different advantages. Therefore it becomes crucial that we expose students to using multiple mathematical representations. This allows students to appropriately choose the representation(s) that best works for the given context (Duval, 2002) and for themselves as learners. This literature review provides a synthesis on the teaching and learning of geometry at the PK-12 level and the role that mathematical representations can play in enriching the geometry experience for our students.

The following research questions guided this review of literature:

(1) *Which frameworks have guided the teaching and learning of geometry in the U.S.?*

(2) *What does the teaching and learning of geometry in the U.S. look like?*

(3) *What is the role of mathematical representations in the teaching and learning of geometry?*

Literature Search Procedures

In order to conduct a thorough review of scholarly literature, an organized search process was used drawing from a variety of databases including EBSCO Academic Search Premier, Education Resources Information Center (ERIC), Science Direct, and JSTOR. The search terms used include "geometry" AND "representations", "geometry curriculum", "geometry curriculum" AND "representations", and "mathematical representations". This initial search resulted in more than 200 pieces of literature which was then narrowed to include only those written only in English. While the authors acknowledge the influence of international perspectives on the teaching and learning of geometry in the U.S., for the purpose of this literature review, any papers discussing this topic in non-U.S. settings were excluded in order to maintain the focus on U.S. PK-12 education. Additionally, any not relating to the previously stated operational definition of mathematical representations for this paper (e.g., articles relating to racial or cultural representations in mathematics) were also excluded as they were deemed beyond the scope of this study. After conducting this search, additional sources were found using the reference lists of each included article. In all, sixty-seven items of scholarly literature consisting of peer-reviewed journal articles and books were included in this paper. The results of this literature search determined the structure of this paper which is organized by the following: (1) frameworks guiding the teaching and learning of geometry in the U.S.; (2) teaching and learning of geometry in the U.S.; and (3) the role of mathematical representations in the teaching and learning of geometry in the U.S..

Frameworks Guiding the Teaching and Learning of Geometry in the U.S.

While the focus of this literature review is to discuss the role of mathematical representations in the teaching and learning of geometry, it is necessary to first consider the frameworks that have influenced geometry instruction. For purposes of this paper, we define frameworks broadly as contributions that are theoretical frameworks, conceptual frameworks, conceptual models, theories, or similar. This section addresses research question one through a discussion of eight frameworks which provide the theoretical background to guide and support research on the teaching and learning of geometry. These frameworks found through the literature search are foundational in understanding how geometry teaching and learning has evolved over time. A summary is provided in Table 1, and details for each framework is provided in this section.

First, Van Hiele's (1986) framework postulates that the five levels of geometric thinking were sequential and hierarchical, and that for students to attain the next level, they must pass through the preceding one. These five levels are visual, analytic, abstract, deductive, and rigor which describe children's levels of thought in learning geometry. While some previous research suggested that these levels accurately describe the development of students' geometric thinking (Burger & Shaughnessy, 1986; Clements & Battista, 1992), in recent decades, researchers are beginning to argue that students may develop their thinking in these various levels simultaneously (Battista, 2007; Clements, 1999). These later researchers maintain that the third framework, abstraction theory (Battista & Clements, 1996), which proposes that learning is a recursive cycle through phases of action, reflection, and abstraction - may reflect a more accurate way to describe students' geometric thinking.

The second framework– the theory of figural concepts (Fischbein, 1993) - attempted to interpret geometrical figures as mental entities that simultaneously possess conceptual and figural properties. According to this notion of figural concepts, Jones (1998) describes that geometrical reasoning is characterized by the interaction between the figural and the conceptual aspects. It is necessary for students to form connections between both the conceptual (abstract) representation and the visual representation, however that is generally where students make the most errors (Jones, 1998).

Table 1

Frameworks for Teaching and Learning Geometry

Name	Description
van Hiele (1986)	Children move through five levels of thought in geometry - visual, analytic, abstract, deductive, and rigor.
Theory of Figural Concepts (Fischbein, 1993)	Geometric figures are mental entities which simultaneously possess conceptual and figural properties.
Abstraction (Battista & Clements, 1996)	The process by which the mind registers objects, actions, and ideas in consciousness and memory, and further describes two forms – spatial structuring, and mental models.
Cognitive Processes (Duval, 1998)	Geometrical reasoning involves three kinds of cognitive processes which fulfill specific epistemological functions: visualization processes, construction processes, reasoning processes.
Geometric habits of mind (Driscoll, 2007)	Teachers need to develop an understanding of geometric thinking and their own geometric habit of mind including: Reasoning with relationships, generalizing geometric ideas, investigating invariants, balancing exploration and reflection.
Concept learning and the objects of geometric analysis (Battista, 2009)	Students need to analyze objects (physical objects, concepts, and concepts definition) and mental entities to understand and reason about mathematics.
Diagrams and representations (Battista, 2009)	Both diagrams, and physical objects play a major role in geometry.
Spaces for geometric work (SGW) (Goméz-Chacón & Kuzniak, 2015)	Describes the work that people (students, teachers, mathematicians, etc.) perform when they solve geometric tasks.

Duval (1998), illustrating the fourth framework, approached geometric reasoning from a cognitive and perceptual lens. He described three cognitive processes which fulfill specific epistemological functions: (1) visual processes which refer to the visual representation of a geometrical statement or the heuristic exploration of a complex geometrical situation; (2) construction processes which refer to the use of various tools; and (3) reasoning processes which refers to the discursive processes for the extension of knowledge, for explanations, and for proofs. He further stated that these processes can be performed separately. In fact, he suggested that these three processes should be developed separately, and that it is necessary to differentiate between them before using them in coordination with one another.

In the fifth framework, Driscoll (2007) shares that teachers need to foster geometric thinking in their classrooms so that students will learn to use geometric thinking as a complement to algebraic thinking in problem solving. He describes that people with mathematical power perform thought experiments, invent things, look for invariants or patterns, make reasonable conjectures, describe things both casually and formally, think about methods, strategies, and processes, visualize things, and seek to explain why things are as they see them. To accomplish this goal, he proposes four geometric habits of mind that teachers need to develop which are: reasoning with relationships, generalizing geometric ideas, investigating invariants, and balancing exploration and reflection. These habits of mind allow teachers productive ways of thinking that enable them to support their students in learning and application of formal mathematics.

The sixth and seventh frameworks, by Battista (2009), further discuss the need for forming concepts from physical objects. He described "geometry instruction and curricula generally neglect the process of forming concepts from physical objects and instead focus on using diagrams and objects to represent formal shape concepts" (p. 97). Often, instruction moves too quickly away from physical manipulatives to diagrams and abstract thinking, or teachers avoid using manipulatives all together, and as a result students often incorrectly connect attributes of a diagram or object to the geometric concept. Students experience a world filled with physical objects, and in order to provide them opportunities to connect mathematics to their world, these physical objects play a crucial role.

In the eighth framework, spaces for geometric work (SGW) explained by Goméz-Chacón & Kuzniak (2015), describes the process that is performed when thinking about geometric tasks. SGW describes two interconnected planes: the epistemological and the cognitive (Kuzniak, 2015). The epistemological plane contains three intersecting elements: (a) real and local space as material support with a set of concrete and tangible objects; (b) artifacts such as drawing instruments or software; and (c) a theoretical frame of reference based on geometric definitions and properties. The cognitive plane (adapted from Duval, 1998) is comprised of three cognitive processes: (a) visualization process connected to the representation of space and material support; (b) construction process determined by instruments (ruler, compass, etc.); and (c) a discursive process which conveys argumentation and proofs. Both the epistemological and cognitive planes are interconnected through the synthesis between three different modes of knowledge: intuition, experiment, and deduction and both need to be articulated in order to ensure complete geometric work (Houdement & Kuzniak, 2003). Although the SGW model was developed for geometry it can also be generalized and connected to other mathematical domains.

In summary, these eight frameworks represent some of the long-standing and current frameworks on geometry teaching and learning. These frameworks lay the necessary foundation for further understanding research on the teaching and learning of geometry in the U.S.. Mathematics is filled with connections between and within domains and the use of mathematical representations allow us to make and leverage these connections. The world students live in is full of shapes with some exhibiting beautiful consistent patterns while others seem to lack symmetry or regularity. Opportunities that allow students to experience the "harmony, beauty, order, clarity, wonder, curiosity, and enjoyment of mathematics" (NCTM, 2020b, p. 15) are important in their development of a positive mathematical identity. While simple formulas are used in school mathematics, they do not account for the irregularity. Therefore, it is crucial for our students to be exposed to the messiness in mathematics that exists in the world around them, and having a toolbox of multiple mathematical representations allows them to make sense of this (Organisation for Economic Co-operation and Development, OECD, 2018). This idea of making important and necessary connections using various types of mathematical representations (i.e. Lesh et. al, 1987; Huinker & Bill, 2017) - visual, symbolic, verbal, contextual, and physical) during geometry instruction will be explored further in a later section.

Teaching and Learning of Geometry in the U.S.

This section addresses research question two to understand why representations as well as connections among representations are essential in PK-12 mathematics classrooms, specifically focusing on both the traditional and current approaches to teaching and learning of geometry. Geometry is one of the oldest branches of mathematics, and its origins can be traced back to a wide range of cultures and civilizations. Yet, the aims and goals of modern geometry instruction are widely debated (Jones, 2000; The Chicago School Mathematics Project, 1971). Jones (2000) states "The fundamental problem in the design of the geometry component of the mathematics curriculum is simply that there is too much interesting geometry, more than can be reasonably included in the mathematics curriculum" (p. 75). At least in North America, in over the past hundred years, high school geometry was comprised of students using Euclid's *Elements* (Sinclair, 2008). In the 1960s, geometry was then explicitly introduced as a topic in primary schools, and focused primarily on the study of two-dimensional geometry to prepare students for Euclidean geometry (ICMI, 1995).

More recent studies claim similar purposes for learning geometry and further extend the purpose of elementary school geometry to focus on spatial reasoning (Clements & Battista, 1992; Battista, 2007), and secondary geometry instruction to focus on dynamic geometry software (Hollebrands, 2003) and connections between geometry to algebraic and symbolic manipulations (Knuth, 2000). Geometry serves as an essential foundation for space and shape, and also draws on elements of other mathematical ideas such as spatial visualization, measurement and algebra (OECD, 2018). "[Geometry and measurement] are among the first mathematical ideas to emerge for young children as they interact with their environment and they deepen through early childhood and elementary mathematics" (NCTM, 2020a, p. 115). Table 2a and 2b include an overview of current geometry standards in the U.S. as denoted by the Common Core State Standards for Mathematics (CCSSO & NGA, 2010), but are similar for many states that have adopted college and career readiness standards. While Tables 2a and 2b show how geometry standards progress across the grade levels in the standards, it is important to consider that geometry is also connected to many other mathematical content domains, and this learning trajectory is a combination of developmental progression and an instructional sequence (as described in Clements & Sarama, 2004) . Mathematics standards are not isolated concepts – they are connected to each other both within and across grade levels. It is crucial for educators to understand these connections so they can link to students' prior knowledge while building a strong foundation for the connections that are still to come (Achieve the Core, n.d.).

Researchers have found that while students in the U.S. are given plenty of exposure to geometry, there is a lack of exposure to deep geometrical thinking, and that many teachers need further development to effectively teach it with depth (Clements, 2003; Driscoll, 2007; Steele, 2013). In fact, this is true among all mathematics domains where a majority of mathematics teachers report that instructional materials given to them provide opportunities to teach major topics addressed by state standards. However, a significantly lower percentage of teachers indicated that their materials addressed these topics with equal time, rigor, and intensity (Opfer et al., 2016). Looking at both *what* and *how* geometry is taught, it becomes evident that most U.S. geometry curricula tends to be scattered and while various topics are taught, much is explored at the surface level and does not support higher levels of geometric thinking (Clements & Battista, 1992; Senk, 1989; Sinclair & Bruce, 2015). This lack of instruction and exposure to deep geometrical thinking is also evidenced by U.S. students' performance on the international level, as evidenced by the TIMSS 2007, 2011, 2015, and 2019 data, which show that geometry has historically been the content domain with the lowest performance, and this is true across all tested grade levels (4th grade, 8th grade, end of high school) (Mullis et al., 2020).

Table 2a

Common Core State Standards - Geometry Standards (K-8)

Grade	Main Ideas
K	- Identify and describe shapes - Analyze, compare, create, and compose shapes
1	- Reason with shapes and their attributes
2	- Reason with shapes and their attributes
3	- Reason with shapes and their attributes
4	- Draw and identify lines and angles, and classify shapes by properties of their lines and angles
5	- Graph points on the coordinate plane to solve real-world and mathematical problems - Classify two-dimensional figures into categories based on their properties
6	- Solve real-world and mathematical problems involving area, surface area, and volume
7	- Draw, construct, and describe geometrical figures and describe the relationships between them - Solve real-life and mathematical problems involving measurement, area, surface area, and volume
8	- Understand congruence and similarity using physical models, transparencies, or geometry software - Understand and apply the Pythagorean theorem - Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres

Adapted from CCSSO and NGA (2010)

Table 2b

Common Core State Standards - Geometry Standards (High School)

Grade	Categories within Geometry	Main Ideas
High School	Congruence	- Experiment with transformations in the plane - Understand congruence in terms of rigid motions - Prove geometric theorems - Make geometric constructions
	Similarity, Right Triangles, and Trigonometry	- Understand similarity in terms of similarity transformations - Prove theorems involving similarity - Define trigonometric ratios and solve problems involving right triangles - Apply trigonometry to general triangles
	Circles	- Understand and apply theorems about circles - Find arc length and areas of sectors of circles
	Expressing Geometric Properties with Equations	- Translate between the geometric description and the equation for a conic section - Use coordinates to prove simple geometric theorems algebraically
	Geometric Measurement and Dimension	- Explain volume formulas and use them to solve problems - Visualize relationships between two-dimensional and three-dimensional objects
	Modeling with Geometry	- Apply geometric concepts in modeling situations

Adapted from CCSSO and NGA (2010)

The Role of Representations in Geometry

Bossé and Adu-Gyamfi (2011) describe six modalities of student learning in geometry – communication, collaboration, reading and writing, real-world examples, multiple representations, and technology. This section addresses research question three in describing the role of multiple mathematical representations in geometry instruction. NCTM (2000) recommends providing students opportunities to select, apply, and transfer among mathematical representations to solve problems. NCTM (2014) describes that one facet of effective teaching is to engage students in making connections among mathematical representations to deepen their mathematical understanding. All students arrive to class with prior formal and informal mathematical experiences, and using multiple mathematical representations allows students to draw on multiple sources of knowledge (Boston et al., 2017). By selecting tasks which allow for students to use multiple mathematical representations, teachers can value and encourage students to draw on their mathematical, social and cultural competence, thereby positioning students as being mathematically competent (Boston et al., 2017; Smith et al., 2017).

Lesh and colleagues (1987) proposed five different types of mathematical representations (i.e. visual, symbolic, verbal, contextual, and physical) which are relevant across mathematical content domains and the importance of making connections between them to deepen students' mathematical understanding. "[Students] will need to be able to convert flexibly among these representations. Much of the power of mathematics comes from being able to view and operate on objects from different perspectives" (NCTM, 2000, p. 361). In 2015, Huinker suggested a consideration to Lesh and colleagues (1987) mathematical representations classification by suggesting that there are two important types of translations that need to be developed: (a) translations *between* these different modes of representations such as translations from a visual model to an equation (adapted from Lesh et al., 1987; NCTM, 2014); and (b) translations *within* a specific mode of representation such as from one visual model to another visual model (e.g. comparing an array and an area model). While research supports the usefulness of representations and the rich mathematical perspectives that representations provide, transferring between and within representations can be challenging for both teachers and students alike. Teachers must be deliberate in creating experiences where students are given the opportunity to make sense of mathematical relationships using multiple mathematical representations (Boston et al., 2017). As students are expected to be flexible translating between and within mathematical representations, it is important for teachers to emphasize this as a part of their daily instruction, in turn influencing students' knowledge and ability to use various representations fluently.

In order to understand the role of mathematical representations that are present throughout the geometry curriculum it is first important to understand the progression of the main ideas in geometry (Figure 1) as shown in the Essential Understanding Geometry series (Dougherty et al., 2014; Goldenberg et al., 2014; Sinclair et al., 2012a, 2012b). Table 3 delves deeper into the big ideas in geometry at the K-2, 3-5, 6-8, and 9-12 level. These big ideas connect to the Common Core State Standards (CCSSO & NGA, 2010) described previously in Table 2a and 2b, and are further examined from a representational standpoint in the grade band subsections that follow.

Figure 1

Progression of Main Ideas in Geometry Based on Work of Dougherty et al., (2014); Goldenberg et al., (2014); Sinclair et al., (2012a), (2012b)

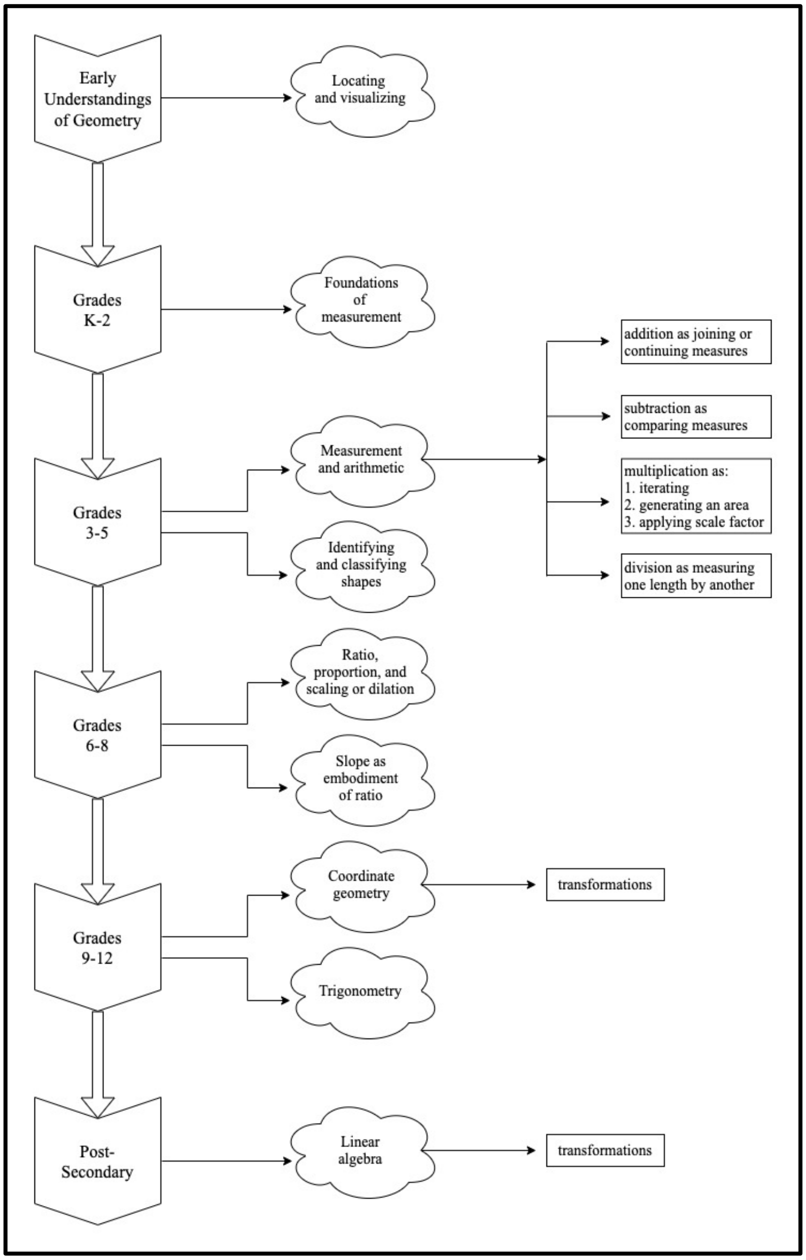

Table 3

Big Ideas in Geometry

Grade Band	Big Ideas
K-2	1: Classification scheme specifies for a space or the objects within it the properties that are relevant to particular goals and intentions. 2: Geometry allows us to structure spaces and specify locations within them. 3: We gain insight and understanding of spaces and the objects within them by noting what does and does not change as we transform these spaces and objects in various ways. 4: One way to analyze and describe geometric objects, relationships among them, or the spaces that they occupy is to quantify – measure or count – one or more of their attributes.
3-5	1: Transforming objects and the space that they occupy in various ways while noting what does and does not change provides insight into and understanding the objects and space. 2: One way to analyze and describe geometric objects, relationships among them, or the space that they occupy is to quantify – measure or count – one or more of their attributes. 3: A classification scheme specifies the properties of objects that are relevant to particular goals and intentions.
6-8	1: Behind every measurement formula lies a geometric result. 2: Geometric thinking involves developing, attending to, and learning how to work with imagery. 3: A geometric object is a mental object that, when constructed, carries with it traces of the tool or tools by which it was constructed. 4: Classifying, naming, defining, posing, conjecturing, and justifying are codependent activities in geometric investigation.
9-12	1: Working with diagrams is central to geometric thinking. 2: Geometry is about working with variance and invariance, despite appearing to be about theorems. 3: Working with and on definitions is central to geometry. 4: A written proof is the endpoint of the process of proving.

Compiled from Dougherty et al., (2014); Goldenberg et al., (2014); Sinclair et al., (2012a), (2012b)

Early Childhood and Elementary Geometry Experiences

From early childhood, the domains of geometry and spatial reasoning are an important area of mathematics learning. Geometry, just as with other areas of mathematics, is an extension of what we do naturally (Goldenberg et al., 2014). Without yet formalizing it, young children are able to understand the distance between themselves and their toys, change location and orientation, and can grasp edges and crawl and run around shapes. In a study involving pre-school participants, Villarroel and Ortega (2017) found that children naturally use geometric shapes in their art even before they have any formal experiences. These early understandings of geometry are supported in the literature (Dougherty et al., 2014; Goldenberg et al., 2014; Sinclair et al. 2012a, 2012b) which indicate that locating and visualizing are students' first introduction to geometry. All these initial exposures to geometric representations engage students in informal reasoning, which support and build a foundation for informal and formal reasoning in K-12 mathematics, and serve as a core in relating other subject areas to mathematics (Clements & Sarama, 2011).

Once students formally start school, students in grades K-2 start to spend time exploring geometry within the context of their own environments and then learn to start engaging in formal activities by identifying and describing the shapes they see and touch (Dixon et al., 2016). In grades 3-

5, students build a foundation of geometric ideas such as dividing shapes into equal pieces which connects to ideas even in high school, such as to trigonometric ratios such as sine, cosine, and tangent (Dixon et al., 2016). By initially forming connections between the visual, physical, and contextual representations, students are then able to develop formal language to describe the shapes. This progression of geometric understanding that students develop at the K-5 level is important to students' overall mathematical learning.

A study by Cai and Lester (2005) in U.S. and Chinese elementary schools found that the types of representations that students use heavily relies on the representations used by their teachers, thereby emphasizing the importance of using multiple mathematical representations during instruction. During a task implementation, Bay-Williams and Fletcher (2017) established that modifying the hundred charts to create an alternative bottom up representation better aligned the concrete and physical manipulatives with the language connected to children's geometrical thinking. Such representations allow for connections to representations that students are exposed to at the K-2 level and beyond, such as physically stacking objects, counting using number lines, and extending to graphing on a coordinate axis. The use of such representations is also supported by Huinker and Bill (2017) who suggest that in these grade levels, visual and physical representations are particularly important as students continue to develop their algebraic reasoning and spatial thinking.

Yu et al. (2009) discuss the idea of prototype and categorical thinking by describing an experiment where students are given visuals of three different rectangles, a vertical, long, and narrow one; a horizontal stout one, and a square. When asked to pick a rectangle, most students pick the horizontal one, as that is the one typically shown in geometry textbooks. This is also seen with students' understanding of other shapes, where a change in orientation often causes much confusion. Children develop their spatial reasoning through both play and focused mathematics instruction, and children's spatial skills strongly correlate to and predict future mathematics performance. As such, this is "an area that that demands greater attention in early childhood and elementary mathematics" (NCTM, 2020a, p. 117). These early experiences of translating between and within these representations are important for students' later understandings of geometric ideas taught at the secondary level which will now be discussed.

Middle and Secondary Geometry Experiences

A central goal of grade 6-8 geometry is to support students in developing a way to talk about properties of shapes, which is consistent with van Hiele's level 3 (Smith et al., 2017). Middle school geometry focuses on examining angles, transformations, congruency and similarity, and the Pythagorean theorem (as described in Nolan et al., 2016). As students transition from elementary to middle school, visual and physical representations should not fade away, but rather need to be developed alongside symbolic representations (Tripathi, 2008). The visual context of a geometry problem plays an integral role in the discovery of number patterns and algebraic expressions, and through the pattern recognition and counting skills developed at the elementary level, and the use of concrete manipulatives, students in middle school can move towards discovering basic geometric formulas (Beigie, 2011). While a focus of middle school geometry instruction is developing formulas, such as those for surface area and volume, these algebraic manipulations naturally lend themselves to connecting the concrete three-dimensional representation. It is important for students to cultivate this conceptual understanding so they can leverage these connections between and within representations and move beyond memorization and rote application (NCTM, 2020b).

High school geometry is often the first opportunity for a formal exploration of inductive and deductive reasoning and proofs. Additionally, high school geometry focuses on making sense of space and visualization as with using transformations, and determining relationships among measurements such as length, area, and volume. The four primary focuses of high school geometry include measurement; transformations; geometric arguments, reasoning, and proof; and solving applied problems and modeling in geometry (NCTM, 2018). "Geometry provides a bridge between many topics in mathematics. It connects functions to their representations, proportions to similar triangles, and triangles to trigonometry" (Nolan et al., 2016, p. 57). Even with these explicit algebraic connections, teachers need to make intentional efforts to connect the symbolic and algebraic to the physical and visual representations that are often brought forward through integrating geometric connections. For example, tasks that allow students to visualize two-and three-dimensional shapes and solids in multiple ways can support conceptual understanding of geometric concepts such as area, surface area, and volume (Ben-Haim et al., 1985; Ferrer et al., 2001) and can enable students to develop further and deeper meaning for these constructs (Smith et al., 2017). Safi and Desai (2017) suggest that teachers can use two- and three-dimensional manipulatives to emphasize connections between algebraic instances—such as multiplying polynomials—with the geometric representations related to the area accounted for through the product of algebraic expressions. Geometric and algebraic understandings and representations reinforce each other, and for students to gain a rich perspective, it is necessary to expose students to both.

In recent years, teachers and students have potentially greater access to new forms of dynamic representations, including open source and freely available dynamic geometry software, virtual manipulatives, and other apps that enable them to manipulate visualizations which was once not possible with the static paper-pencil methods (Hollebrands & Dove, 2011; Jackiw, 2001). As described by Battista's (2009) framework, briefly described in Table 1, such representations allow students to connect conceptual knowledge to dynamic pictorial representations, thereby providing rich opportunities for understanding and connecting geometric representations. Hollebrands (2003) recommends that teachers use dynamic geometry software to support students in gaining deeper understanding of transformational geometry concepts and the connections between transformations and functions. Dynamic software applications introduce students to mathematics that would have otherwise been out of reach and help students transfer metal images of concepts to visual interactive representations that can lead to more robust understanding (Dick & Hollebrands, 2011). Much of secondary mathematics focuses on formal and rigorous mathematical reasoning, and oftentimes there is a greater emphasis on algebraic or symbolic manipulation and logical deductions (Battista, 2017). While this emphasis is indeed necessary, it is equally important for students to be given experiences with other forms of representations (both static and dynamic) to build their initial conceptions of the topic. Such explorations allow for connecting multiple mathematical representations while providing affordances from each representation that can be leveraged in future mathematical explorations.

Concluding Remarks

This literature review synthesizes, organizes, and elaborates on existing literature relating to the teaching and learning of geometry at the PK-12 level in the U.S. and the role that mathematical representations can play in enriching the geometry experience for our students. Through this synthesis, it is evident that representations play a crucial role in the teaching and learning of geometry. By providing students access to opportunities to explore multiple mathematical representations, they are no longer limited by the strengths and weaknesses of one particular representation (Elia et al., 2007), and they are able to deepen their mathematical understanding while engaging in meaningful mathematical discourse (Lesh et al., 1987; NCTM, 2014). Yet, as NCTM (2020b) discusses, at the early childhood and elementary level "Geometry instruction, typically, does not move beyond shape names or definitions, only engaging in low-level thinking" (p. 119). As evidenced within the geometry curriculum, this is similar at the secondary level where algebraic and symbolic representations are greatly overemphasized (Knuth, 2000). As a result, students often experience a disconnect in transferring between and with representations because some representations, especially symbolic and visual, are included as end products rather than as starting points in reasoning and problem solving. "Children enter this world as emergent mathematicians, naturally curious, and trying to make sense of their mathematical environment" (NCTM, 2020a, p. 17). For our students to continue to see themselves as capable learners and doers of mathematics and experience the wonder, joy, and beauty of doing mathematics, it is important that PK-12 instruction provides them opportunities to see connections between mathematics and their daily lives (NCTM 2018, 2020a, 2020b).

Giving students such opportunities to engage in tasks that allow the use of multiple mathematical representations empowers teachers to create more equitable tasks as they afford a wider range of access to mathematical ideas (Boston et al., 2017). However, the use of multiple mathematical representations is often placed into the curriculum as an afterthought to help students who may be struggling to firmly understand the content. Van de Walle, Karp, and Bay-Williams (2019) describe understanding as existing along a continuum from instrumental to relational understanding, terms that were first introduced by Richard Skemp in 1976. While students may perform well academically in the moment, as teachers it is important to think *about* and *through* whether students are just remembering or whether they are thinking about the mathematics. Clements (2003) supports the idea that all students need to play with concrete objects and see visual representations before they are able to understand abstract topics. Students get little meaningful mathematics out of the traditional proof-based approach that is often used in the high school geometry curriculum. Some students may be able to remember and give an output in the given amount of time, but "if we look at the mathematics in the world and the mathematics used by mathematicians, we see a creative, visual, connected and living subject" (Boaler, 2016, p. 31). Geometry naturally lends itself to noticing and wondering about the world around us and provides an ideal platform to make these representational connections a reality. Consistent PK-12 geometrical learning experiences through high quality tasks rightfully affords students intentional opportunities to translate between and within multiple mathematical representations, empowering students to experience this wonder, joy, and beauty in mathematics (NCTM 2018, 2020a, 2020b). In this manner, students' experiences can be fueled by discovering connections between and within mathematical representations, while linking mathematical domains enriched by the wonders of geometry in our lives, communities, and cultures.

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Impact of a Summer Camp on Elementary Students' Understanding and Awareness of Engineering Careers and Attitudes toward Engineering

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ABSTRACT

Students who develop interest in STEM careers by the eighth grade are more likely to pursue careers in STEM (Tai et al., 2006). Interest development can happen through a variety of sources, including informal learning experiences such as out of school programs and summer camps. This study looks at one such informal STEM experience, an engineering summer camp for elementary students, to explore how this camp impacted their understanding, awareness, and attitudes toward engineering. The study used a pre/post design to determine the impact of the camp with two groups of students in two separate years. The results suggest that students gained an awareness of the types of engineering, a better understanding of the purpose of the work of engineers, and had more positive attitudes about the value of engineering and their own aspirations toward engineering.

Keywords: elementary engineering education, outreach, camps, attitudes

Introduction

There is a growing need to develop a workforce that is capable of meeting the needs of a changing society. The World Economic Forum (2017) recently published a report indicating that the workforce of the future will be heavily dependent on technology and engineering. The number of science, technology, engineering, and mathematics (STEM) jobs in the United States is projected to grow at a rate 4.6% higher than non-STEM jobs between 2019 and 2029 (U.S. Bureau of Labor Statistics, 2020). While the number of STEM graduates in the U.S. is steadily rising, a growing number of those graduates do not remain in the U.S. workforce, instead returning to home countries such as China and India (National Science Board, 2018). This points to a need for more US students to graduate from STEM programs to meet the job demand, which will require attracting students to STEM majors prior to entering college. A growing body of research indicates that early experiences in and out of school are effective at developing STEM interest, which in turn influences students' choices to pursue STEM careers (Maltese et al., 2014; Maltese & Tai, 2011; Sadler et al., 2012).

Interest develops at all ages; however, students are more likely to pursue a STEM degree if they develop some interest in a STEM career prior to middle school (Tai et al., 2006). This interest can develop through a variety of sources. Many people who pursue a STEM degree attribute their interest to early life experiences with a family member or friend who is an engineer or scientist, opportunities to tinker with devices or play outside, or a natural interest in the subjects (Maltese et al., 2014). Learning experiences in elementary school can also play a vital role in developing interest and understanding of the field of engineering (Burt & Johnson, 2018). Thus, students, particularly those who do not have family or community support for STEM, need opportunities to experience engineering in order to gain exposure and understanding of what it entails. Formal learning experiences in the school are the most logical setting to gain this exposure, but this is not always possible. Many elementary school teachers lack the confidence to appropriately teach science and engineering in the classroom, causing them to leave these subjects out of instruction or to insert ineffective activities into the curriculum (Appleton, 2013; Yoon et al., 2011). Consequently, informal learning experiences can offer students learning opportunities that may be limited in formal K-12 settings.

Even though we know STEM interest develops early, the research on this development among elementary students is limited. Much of the current research on early learners focuses on conceptions of an engineer (Pekmez, 2018; Newley et al., 2017; Capobianco et al., 2011; Karatas, Micklos, & Bodner, 2011) or identity (Capobianco et al., 2012; Capobianco et al., 2017). While these are important factors that play a role in students' future career decisions, they do not paint the whole picture. This is particularly true when students do not have accurate representations of engineering careers which could limit their ability to see their possible selves (Markus & Nurius, 1986) in those future roles. This study attempts to address this gap in the literature by focusing on understanding of engineering alongside career awareness and interests of upper elementary students.

This study examines an engineering summer camp for elementary students. The goals of the camp were to provide engineering experiences for students in grades 4-6, teach them about the process of engineering design, and expose them to a variety of engineering fields and career options. We pursued each of these goals to provide students with experiences that might enhance their ability to envision engineering as a future possible career path. The purpose of the study was to explore how this camp impacted elementary students' understanding, awareness, and attitudes toward engineering. The study addressed the following research questions:

1. How did the engineering camp affect student understanding of the work of engineers and awareness of engineering careers?
2. How did the engineering camp affect students' attitudes toward engineering?

Background Literature

To guide our study related to students' interest in and awareness of engineering careers, we reviewed literature on career interest development and attitudes toward engineering. Additionally, we explored research focused on student understanding of engineering and in particular their understanding of the work of engineers.

Interest Development and Career Awareness

The development of interest in engineering and engineering careers is a critical aspect of students' choice to pursue those careers. The development of this interest can happen in a variety of ways. Positive learning experiences both in and out of school, and students' beliefs and attitudes about engineering play an important role in this development (Banerjee et al., 2018; Burt & Johnson, 2018; Dou et al., 2019; Lent et al., 1994; Maltese & Tai, 2011). Several studies have pointed out that STEM interest development happens most often prior to middle school (Maltese et al., 2014; Tai et al., 2006; Wyss et al., 2012). A critical age for the development of these interests occurs in middle childhood, particularly in upper elementary school where children begin to think of their interests and capabilities as becoming solidified (Todt & Schreiber, 1998). Students also need guidance to understand that achievement and success are not strictly innate, as well as assistance with developing a growth mindset (Harter, 2006).

The development of interest in engineering and engineering careers can be increased through a variety of mechanisms. One study examined the influence of a design and build workshop with students in upper elementary and middle school. The results indicated that the experience had a positive impact on students' attitudes toward engineering, views that engineers are problem solvers and impact the world, and familiarity with engineering. Additionally, students' self-efficacy and interest in STEM increased after the intervention (Innes et al., 2012). Sullivan and Bers (2019) studied the effects of a robotics program on the interests and attitudes of early elementary students. They found that boys had a higher initial interest in engineering than girls, but after the intervention, girls' interest rose significantly. These authors also noted that the teachers for the robotics program were all female, perhaps providing positive examples for the girls in the study. Ozugul et al. (2017) examined the engineering knowledge and interest of students in grades K-5, finding that understanding of and interest in engineering were not significantly different between males and females, but Caucasian students had significantly higher knowledge and interest levels than Latino/a students. The disparity between ethnic groups can arise when students from one group have limited access to opportunities that promote STEM. These authors suggested that in order to decrease this disparity, engineering interventions should begin in early elementary school. Furthermore, they suggest these interventions should continue through upper elementary school to avoid the split in knowledge and interests in genders that commonly occurs as students progress through secondary schools.

Studies of elementary students in engineering camps are lacking, supporting the need for this study. However, there are several studies that indicate positive effects of engineering camps on interest and understanding of engineering for middle and high school students. Mohr-Schroeder et al. (2014) ran a one-week camp for middle school students that provided students with a variety of hands-on engineering and science experiences led by college faculty and local teachers. They found that their camp was engaging to participants and reported a positive change in career interest between a pretest and posttest. Yilmaz, et al. (2010) found that their camp for high school students improved engineering career interest through an interdisciplinary approach using a variety of hands-on engineering projects. These projects involved real-world scenarios and challenges where students worked in teams to complete the challenges over the course of the week-long camp. Furthermore, Kong, et al. (2013) surveyed over 1,500 middle school students from eight different schools and found that when controlling for initial interest, those who had participated in a science- or engineering-based camp were more likely to want to pursue an engineering career.

Understanding of Engineering

Engineering is a field that involves the design and support of systems and objects that make the world more efficient and productive (Trevelyan & Williams, 2019). Engineering affects every area of life and has a major impact on people of all ages. However, when asked to describe the work of an engineer, many elementary students have misconceptions or lack understanding of engineering altogether. Typical responses from students depict engineers as mechanics, fixers, or laborers, without acknowledging the role of design in the work of an engineer (Capobianco et al., 2017; Capobianco et al., 2011; Gibbons et al., 2004; Newley et al., 2017; Reeping & Reid, 2014). One reason for these misconceptions about the work of engineers is that many elementary teachers feel unprepared to teach engineering and often do not present these experiences regularly (Banilower et al., 2013), in spite of integration of engineering practices into the *Next Generation Science Standards* (NGSS Lead States, 2013) and many states' standards (Lopez & Goodridge, 2018). Therefore, the primary source of information about engineering is developed from portrayal in media or connections with engineers who are friends or family (Bevins et al., 2005; Chou & Chen, 2017; Jacobs & Scanlon, 2002). One of the issues this presents is that many students do not have any personal connections to engineers and are therefore only exposed to limited, and often inaccurate, examples of engineers. Rao and Dewoolkar (2021) examined the portrayal of engineers in the news media and found that engineers were rarely mentioned and news stories missed out on opportunities to describe engineers as experts capable of helping solve key problems. Ellestad (2013) studied popular media, finding that engineers are often portrayed as socially awkward, white males, and with exaggerated and farcical characteristics. The study found that the media furthered people's stereotypical images of engineers, even when personal experiences with engineers countered these images.

Understanding student conceptions of the work of an engineer is not always easy to do. One common method is the use of drawings to gain insight into students' perceptions of the work of an engineer. Capobianco, et al. (2011) conducted a study in which they collected data from nearly 400 elementary students in various types of schools in the Midwest. The authors demonstrated that these students often depicted engineers as mechanics, laborers, and technicians. They also found that more than half of the drawings portrayed engineers as men. Middle school students have also been found to share conceptions of engineers as predominantly males who are portrayed as makers (Fralick et al., 2009; Hammack & High, 2014; Karatas et al., 2011).

Interventions have had some success in developing an understanding of the work of engineers in students. Farland-Smith and Tiarani (2016) looked at two groups of eighth grade students, one in a traditional science classroom setting and another in which the teachers brought in engineers from the community and focused on engineering careers and the integration of STEM subjects. While both cohorts had similar misconceptions initially, the integrated STEM cohort developed a more comprehensive understanding of the work of an engineer and how an engineer uses science. Hammack et al. (2015) studied the effect of an engineering-focused summer camp on the understanding of engineering, determining that participants gained a better grasp of engineers as being involved in the design and development of products. Similarly, Hammack and High (2014) examined the impacts of an afterschool engineering mentoring program for middle school girls and found that prior to participation, students viewed engineers as people who build and fix things. After participating in the program, the girls were more likely to portray engineers as creative problems solvers.

Attitudes Toward Engineering

Elementary school students' attitudes toward engineering vary based on a variety of factors, though there is some indication that attitudes are less positive in elementary school and more positive in secondary schools (Kőycű & de Vries, 2016). Additionally, there are studies that indicate differences over whether males or females have more positive attitudes toward engineering in elementary school (Lachapelle & Cunningham, 2019; Lie et al., 2019). However, studies tend to support the notion that interventions focused on engineering education improve student attitudes toward engineering. Baran, et al. (2016) looked at an elective weekend out-of-school STEM program for middle school students. While students generally had positive attitudes starting the camp due to its voluntary nature, the program still made significant improvement to attitudes, particularly in regard to personal and social implications for engineering, use of science in engineering, and daily-life connections for STEM subjects. Another study, by Teeter, et al. (2020), examined an engineering-focused exhibition for high school students. They suggested that outreach events such as these, which focus on developing interest and engagement, improved attitudes toward engineering in conjunction with the development of identity toward engineering.

Student exposure to engineering in the classroom and informal experiences can help them to identify more closely with engineers. Kelly, et al. (2017) describe this development as a progression, from simply seeing a person acting in the role of an engineer, to actually doing the work of an engineer themselves. A study by Douglas, et al. (2014) examined students in grade 2-4 classes with teachers who had taken professional development in engineering education. The teachers implemented engineering lessons in their classes throughout the school year. Researchers found that students identified themselves as engineers more after the intervention than they had before. Another study of middle school students reported that integration of engineering into the curriculum increased students' identity as engineers (Yoon Yoon et al., 2014).

Theoretical Framework

The possible selves framework guided our work on this study. Possible selves are the ideas individuals possess about what they might become (Markus & Nurius, 1986). These possible selves include visions of who they would like to become as well as who they fear they might become. Possible selves are separate yet connected to current and past selves and are distinctly social. Perceived possible selves can be the direct result of how individuals compare themselves with the characteristics and behaviors they have witnessed of others. "What others are now, I can become" (Markus & Nurius, 1986, p. 954). While individuals might have multiple possible selves, the possible selves they create are derived from their personal experiences and exposure to different models, images, and symbols.

For children, career related possible selves can be ideas about what is possible for their future. These ideas of possible careers can then guide their behavior to help them achieve a desired outcome. Children may perceive certain possible selves to be more achievable than others based, in part, on exposure to the career and availability of role models (Oyserman et al., 1995). It is only when a hoped-for self seems possible that a child attaches certain actions to that self and can envision a path toward achieving that possible self (Oyserman & Markus, 1990). This means having limited exposure to accurate representations of different careers, such as engineering, can limit a child from developing the possible self of engineer. As such, one of our primary goals when designing the camp experiences for children was to provide them with experiences that connected the camp activities they completed with the work of real engineers. By providing campers with opportunities to learn about the work of engineers from the engineers themselves, we hoped to provide them with opportunities to envision a possible self where they could become an engineer.

Methods

The possible selves framework (Markus & Nurius, 1986) guided the design of the study intervention and our interpretation of the findings. Researchers sought to evaluate the effectiveness of the engineering summer camp intervention for elementary students by using a one-group pretest-posttest design (Strunk & Mwavita, 2020). Prior to any data collection, IRB approval was gained for this evaluation project. Thus, for each participant, researchers gained student informed consent and parent consent prior to participant data being included in this study.

This study utilized a purposeful sampling strategy that required the researchers to establish criteria for the sample prior to data collection (Hays & Singh, 2012). Further, Miles and Huberman (1994) suggest that there are a variety of purposeful sampling strategies. Based on the literature suggesting that middle childhood is a critical time for interest development, we purposefully recruited middle childhood participants for our camp. Through the selection of all participants in the summer engineering camp across two years of the camp, a comprehensive sampling method provided the most representative sample.

Participants

Camp participants (n = 49) came from two summer camps held in consecutive years that were organized by the study authors and included elementary students who were going into grades 4-6 the next academic year. Only 44 camp participants were included in data analysis as five were lacking either a complete data set or parental consent to be included in the study. Table 1 provides an overview of the study participants' demographics by camp year. The students were between the ages of nine and 12 with about two-thirds being male. In terms of race, 55% were white and 25% identified as American Indian or Alaska Native. The remaining students identified as Asian, Hispanic or preferred not to respond. One student was present for camp in both summers, and no other student had participated in this camp before.

Table 1

Demographics of Camp Participants

	Year 1[a] n (%)	Year 2[b] n (%)
Gender		
Male	16 (70%)	14 (67%)
Female	7 (30%)	7 (33%)
Race/Ethnicity		
White	10 (43%)	13 (62%)
American Indian/Native Alaskan	8 (35%)	3 (14%)
Asian	2 (9%)	4 (19%)
Hispanic/Latinx	1 (4%)	0 (0%)
No response	2 (9%)	1 (5%)

[a]Year 1: n = 23
[b]Year 2: n = 21

Instructional Context

The camps were five days in length and held on a large university campus in a Midwestern state, where local elementary students were invited to participate in learning about engineering design. Each day of the camp ran for 3.5 hours, with snacks and a break provided in the middle of each day. The camp has been operating at this university since 2016 and is facilitated by faculty and graduate students in the mathematics and science education program. The goals of the engineering camp were to (1) introduce students in grades 4-6 to the engineering design process and how it can be used in the context of solving a problem, (2) help students develop an understanding of the work of an engineer, and (3) increase engineering career awareness by presenting a variety of engineering fields and careers associated with those fields.

On the first day of camp, the students were divided into teams of 3-4, introduced to the camp staff, and completed pretests. The camp director then provided an overview of engineering design and how the campers would use it throughout the week. The remainder of the week was spent completing engineering design tasks, touring engineering facilities on campus, learning about engineering careers, and talking with engineering faculty about their various fields of study. Camp instructors used a variety of strategies to ensure all learners were actively engaged and able to access the material, including but not limited to, small group work, hands-on modeling of abstract concepts, using probing questions to elicit student thinking and facilitate meaning making, and exposure to gender and ethnically diverse professional engineers. At the end of the week, campers completed posttests and the campers' families were invited to come to a showcase event, where the projects from the week were displayed and campers shared their experiences at the camp with invited guests. Table 2 provides an overview of each day's schedule.

Table 2

Camp Schedule Overview

Day 1	Days 2-4	Day 5
• Pretests • Introduction to engineering design • Design challenge • Exit ticket • Take-home challenge	• Engineering career highlight activity • Design challenge • Guest speaker or facility tour • Design challenge • Exit ticket • Take-home challenge	• Engineering career highlight activity • Design challenge • Posttests • Family showcase

The possible selves that children view as achievable are influenced by their personal exposure to careers and role models. With this in mind, we structured each day to highlight specific engineering fields and careers through in-person visits from engineering faculty and videos that showcased engineers at work. Camp staff solicited engineering faculty volunteers who were both interested in speaking with elementary students and whose schedules allowed them to visit camp in person. This allowed campers to see, hear, and pose questions to both male and female engineers in fields such as mechanical, aerospace, chemical, civil, architectural, biomedical, environmental, agricultural, and electrical engineering. The design challenges for the camp were selected with several criteria in mind. First, the developmental appropriateness of the lesson was considered to ensure elementary friendly activities were chosen. Second, chosen activities represented good principles of engineering design as laid out in *A Framework for K-12 Science Education* (National Research Council, 2012). Many of these tasks were taken or adapted from sources such as TeachEngineering (http://teachengineering.org) and Engineering is Elementary® (Museum of Science Boston, 2003). Third, the types of design challenges were selected to align with the career fields of the visiting engineers. Different design challenges were used at each camp in order to keep the camp engaging for students who might attend multiple years.

Measures

Researchers used several instruments to assess the impact of the camp on students' understanding of the work of an engineer, potential engineering career paths, the engineering design process, and attitudes towards engineering. Details about each instrument are provided below.

What is Engineering? (WiE)

This instrument, developed by Cunningham (2005), explores student conceptions about the work of an engineer. It was developed to allow researchers to gain insight into the depth of knowledge students have about engineering. This instrument consists of sixteen images and descriptors along with one free response question. Image descriptors include improve machines, construct buildings, arrange flowers, and read about inventions. Students are asked to identify the items that represent the work that engineers engage in, and then describe an engineer in words. The What is Engineering? instrument has been shown to have good internal consistency with a Cronbach's α of 0.881 (Cohen, 1988).

Engineering Design Process Questionnaire (EDPQ)

The EDPQ instrument explores student understanding of the engineering design process and the work of an engineer. It includes three open-ended questions: (1) describe the work of an engineer, (2) list as many different types of engineers as you can and describe the jobs that each might have, and (3) describe the engineering design process. There is also a Likert-style question where students rate their understanding of the engineering design process, ranging from not knowing at all to understanding very well. For this study, only the first two questions were analyzed. This instrument was developed by the researchers for the purposes of the camp and to address the research questions for this study. To ensure that the questions were appropriate for upper elementary school students, the researchers determined their Flesch-Kinkade readability score, which indicated a 5th grade reading level. The camp instructor read questions aloud for those participants who required additional assistance.

Engineering Interest and Attitudes (EIA)

The EIA instrument was developed by Lachappelle and Brennan (2018) to determine the extent to which students develop interest in engineering and what their attitudes are toward engineering upon encountering engineering design. This instrument consists of a twenty-four item five-point Likert scale in which students rate their agreement to each item from strongly disagree to strongly agree. Items are grouped into subcategories to evaluate engineering attitudes among students according to the value to me, enjoyment, value to society, school engineering, aspirations, and gender bias. This study was looking at the effect of the camp on students' understanding of the work of engineers, awareness of engineering careers, and attitudes toward engineering. Based on the research purpose and the nature of the intervention, the researchers chose to eliminate the subcategories for school engineering and gender bias. The instrument, as designed, asks for students to assess themselves in their past and in the present. The instrument was modified for this camp to record only their current attitudes and interests in the pretest and posttest. To validate the EIA, Lachappelle and Brenna (2018) established the validity of the instrument using both content validity via an expert panel and through both exploratory and confirmatory factor analysis.

Data Collection and Analysis

Camp facilitators gave each instrument as a pretest to the campers on the first day of camp, then retested them on the last day of camp. Each student completed a demographic questionnaire, followed by the WiE, the EIA, and the EDPQ. Quantitative data was transferred to SPSS for analysis. The first author scored the first two instruments for each student by indicating whether each item was correctly or incorrectly selected and found the total number of correct selections for each student.

The pre-post data did not meet the assumption of normality; therefore the data was analyzed using the Wilcoxon Signed-Ranks test. The EIA was divided into subscales by averaging each of the individual item responses that make up each subscale. These averages, which also did not meet the assumption of normality, were run using the Wilcoxon Signed-Ranks test.

The open-ended responses from the EDPQ and WiE instruments were transcribed into Excel and analyzed independently by the first two authors. Instead of establishing an a priori coding system prior to looking at the responses, the authors used the phrases given by the participants to establish the coding (Saldaña, 2015). The responses to the first question were read twice to look for keywords and responses that the participants used to describe the work of an engineer, such as improve, fix, and design. Researchers then developed codes from the key words and phrases, lumping those with the same meaning and context. The frequencies of these codes were then measured for both pre and posttest. Once complete, the researchers met to discuss their findings and compare notes. During this meeting, any discrepancies in the way the phrases were coded were discussed until a consensus was reached. Once the researchers agreed on the coding, they ensured that frequencies for each code matched for analysis. Researchers then looked at each participants' pretest and posttest response to compare the two and search for meaningful changes in the responses.

The engineering type questions were marked and labeled for each type of engineering that was identified by the student, classifying it first as accurate or inaccurate, then within a subcategory of the type of engineering. Accurate responses correctly identified engineering types, such as mechanical, civil, and chemical. Inaccurate responses included descriptions of actions like "repair cars" or jobs that are not engineers like "plumber". The responses were counted and compared graphically, then analyzed using a Wilcoxon Signed-Ranks test.

Results

What is Engineering? Instrument

Figure 1 displays the results of the WiE instrument, ordered from largest decrease to largest increase. The items "Design Ways to Clean Water" and "Work as a Team" had the largest increase, while "Improve Machines" and "Design Things" remained the most selected items on the posttest. Table 3 displays the results of the Wilcoxon Signed-ranks test, which indicated that student conceptions of engineering before camp (*Mdn* = 10) were not significantly different from their conceptions after camp (*Mdn* = 11), $Z = -1.799$, $p = .072$. After discussion between the authors about the reason for the lack of a significant increase in scores and an examination of the data, it was determined that many students selected the item stating that "engineers teach children" in the posttest. During both weeks of camp, engineers came to speak to the students about their discipline, and the authors found it likely that students associated this as part of the work of an engineer. This item was subsequently removed from the data set and the test was run again. Without the teaching item included, the test indicated a significant difference between the pretest (*Mdn* = 9) and posttest (*Mdn* = 10), $Z = -2.324$, $p = .02$. The effect size ($r = .35$) for this analysis was found to be small according to Cohen's (1988) convention.

Engineering Design Process Questionnaire

The EDPQ instrument included a question that asked students to describe the work of an engineer, and the WiE instrument asked participants to complete the following prompt: "An engineer is a person who…". These questions were very similar and were given within a few minutes of each other, but the responses were not always the same. For example, in the pretests participant 45 described an engineer as a person who "solves problems", but described the work of an engineer as "fixing things (electronics, etc.)". Additionally, in the posttests, participant 41 stated that an engineer "makes the world a better place", but described the work of an engineer as "buildings, vehicles, chemical, farms, human body". Overall, approximately 40% of the participants provided answers that were different on the two questions.

Figure 1

What is Engineering? Pretest/Posttest Responses

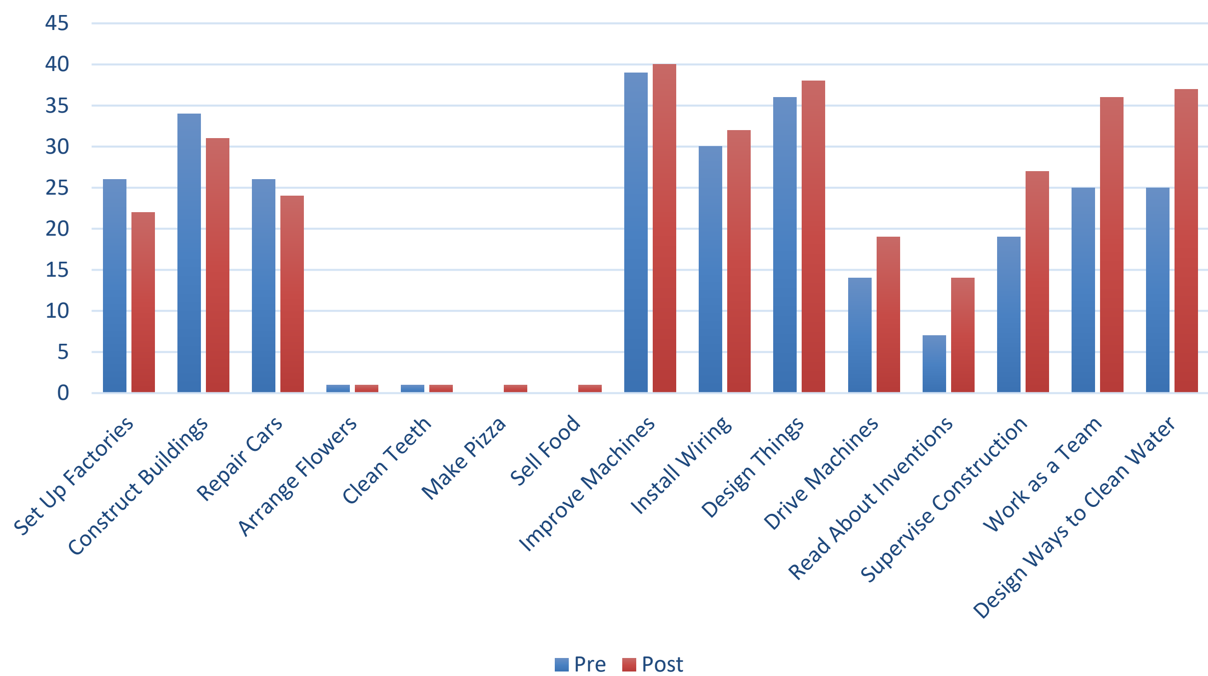


Table 3

What is Engineering? Pretest/Posttest Results

	Pretest			Posttest					
Subscale	Min	Max	Mdn	Min	Max	Mdn	*Z*	*p*	*r*
What is Engineering?	6	14	10	8	16	11	-1.799	.072	.27
Modified What is Engineering?	6	13	9	7	15	10	-2.324	.020*	.35

*Indicates significance at $p < .05$

The responses were compiled from both instruments, and the responses varied among the participants. The most common responses on the pretest indicated that engineers design (28 instances), build (27 instances), and repair or fix (17 responses). Eleven participants left their response blank or stated that they did not know. All other responses were recorded 10 or fewer times. On the posttest, the top two responses remained prevalent, with design occurring 24 times and build occurring 17 times. However, three categories increased by a large margin from pretest to posttest. "Problem solving in engineering" increased from 10 responses on the pretest to 23 on the posttest, making it the second most-used phrase. "Helping" increased from 3 to 16, and "improve" increased from 10 to

16. Furthermore, the number of responses from students that said they did not know dropped from 11 on the pretest to 6 on the posttest.

Examination of individual responses before and after the camp revealed few major differences for students, except for the addition of clarifying or purpose statements. For example, participant 13 stated in the pretest that "engineers usually design and fix things", while their posttest response stated that "engineers usually try to make society better by making new things and improving old things". Participant 9 began with the idea that engineers "improve and plan and build stuff", but after the camp stated that "they help improve our lives".

The second question on the EDPQ asked students to name as many types of engineers as they could. Figure 2 displays the results, which indicated that the number of correct responses increased from 30 to 133, the number of incorrect responses decreased from 26 to 4, and the number of students who did not respond decreased from 12 to 7. To ensure that the increase in number of correct responses was not due only to one or two students who were able to name a large number of engineers, the responses were analyzed for each student, and categorized into responses that gave zero correct responses, one to three correct responses, and four or more correct responses. The results, shown in Figure 3, indicate that the number of students naming zero engineers dropped from 27 to 6, the number naming one to three engineers remained the same, and the number naming four or more engineers increased from 1 to 21.

Finally, researchers analyzed the change in total number of correct responses by running a Wilcoxon Signed-Rank test, which indicated a significant increase in number of engineers the students could name, $Z = -5.040$, $p < .001$, and revealed a medium effect size ($r = .56$) which indicates that the camp experience was effective at increasing students' awareness of an existence of different engineering disciplines.

Figure 2

Total Number of Correct Types of Engineers in Participants' Responses

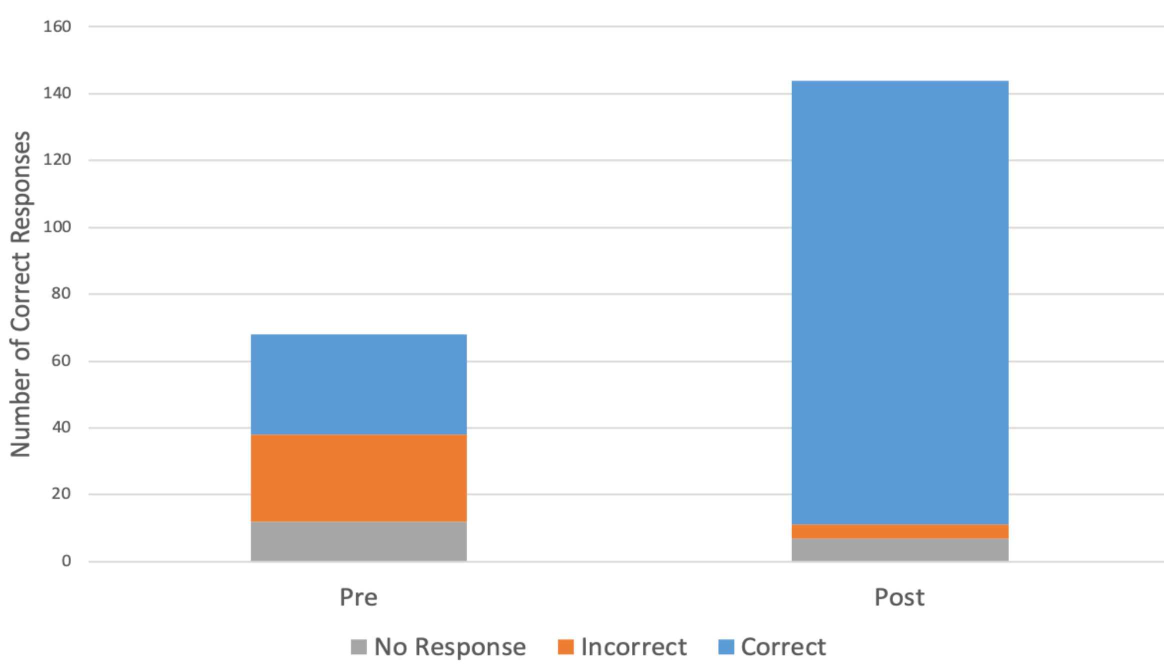
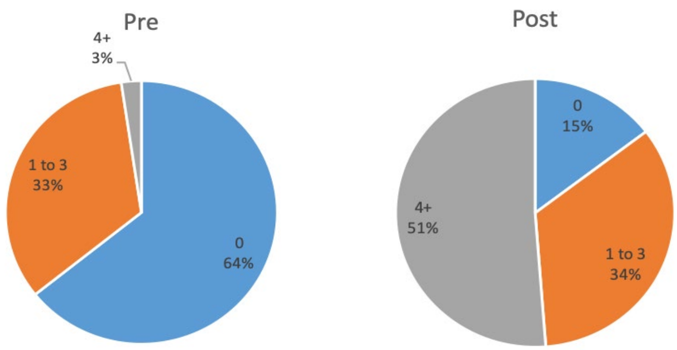


Figure 3

Number of Correct Responses Given by Individual Participants



Engineering Interests and Attitudes Scale

Table 4 presents the results of the Wilcoxon Signed-Rank test for each of the EIA subscales. The EIA instrument indicated that participants' attitudes in each of the subscales either increased or remained the same, however only two categories were significantly different from pretest to posttest. The test provided this difference for the value of engineering to society ($Z = -3.782$, $p < .001$) and aspirations toward engineering ($Z = -2.284$, $p < .022$). The effect size for the value of engineering to society ($r = .59$) suggests a medium effect, and aspirations toward engineering ($r = .36$) suggests a small effect. The value of engineering to me subcategory, while not significant at $p < .05$, also revealed a small effect size ($r = .30$). According to the convention developed by Cohen (1988), these effect sizes suggest that the camp was somewhat effective at improving students' attitudes toward engineering in these subcategories. However, caution should be used when interpreting the meaning of effect sizes because these general categories developed by Cohen may be interpreted differently according to the context in which they are used.

Table 4

Engineering Interests and Attitudes

	Pretest			Posttest					
Subscale	Min	Max	Mdn	Min	Max	Mdn	*Z*	*p*	*r*
Value of Engineering to Me	2.0	5.0	4.0	2.0	5.0	4.5	-1.919	.055	.30
Value of Engineering to Society	2.6	5.0	3.8	1.0	5.0	4.8	-3.782	<.001**	.59
Enjoyment of Engineering	1.0	5.0	4.0	1.0	5.0	4.0	-1.287	.198	.20
Aspirations Toward Engineering	1.0	5.0	3.9	1.0	5.0	4.5	-2.284	.022*	.36

*Indicates significance at p < .05
**Indicates significance at p < .001

Discussion

The analysis of the WiE instrument suggested that camp participants initially believed that engineers engaged in tasks such as repairing cars and constructing buildings, which is in line with prior work on understanding of engineers (Capobianco et al., 2017; Capobianco et al., 2011; Newley et al., 2017). After analysis, the results from the pretest to posttest were not significant. However, the removal of the item regarding teaching did make the test significant, and the effect size increased from $r = .27$ to $r = .35$. While this effect size is considered small according to Cohen's convention, the growth in understanding of the work of an engineer is consistent with other engineering interventions (Farland-Smith & Tiarani, 2016; Hammack et al., 2015). However, seeing that one aspect of the camp so prominently affected the results of the posttest, we feel that it is necessary to consider the importance of the context of an intervention in determining the results. These results indicate that the way in which curriculum is presented can have a substantial impact on the way students view or understand a particular topic. Specific aspects of the curriculum that are either included or left out may play a meaningful role in participants developing an understanding of what engineers do, and should be an important consideration for the design of future experiences.

Another consideration to be made based on the results of this study is the need for multiple assessment methods in research. There were similar questions about the work of engineers on two separate instruments, and 40% of participants responded with different answers. While for some, these differing responses may represent the variety of concepts they have about engineers, it also demonstrates the fragility of these participants' understanding of engineering. By using two measures, it was possible for the researchers to gain insight into the variety of ideas that participants had, while also seeing how their understanding lacked depth and stability.

Nevertheless, certain shifts in the responses did indicate that the context, message, and activities present in the camp had an impact on participants. The biggest increases in students' responses to describing engineers and their work involved engineers helping, improving, and finding solutions. Many of the participants specifically included these aspects of the work of engineers into their responses alongside other practical activities such as designing and working as a team. Typically, the engineers that visited in person or were displayed on video talked about the impact of their work on people's lives. Additionally, many of the challenges were rooted in real-world problems and described how engineers could work to solve the problems at hand. This change in participant responses represents a deeper understanding of what engineers do and why they do it. The participants understand that there is a purpose to the work of engineers. Such changes suggest that the messages from camp speakers, videos, and challenge scenarios may have influenced students to see engineering as helpful to the world around them.

Results from the career awareness section of the EDPQ were analyzed in two different ways to determine participants' ability to name different types of engineers. The first analysis demonstrated that the total number of correct responses increased dramatically, and the number of incorrect responses and no responses decreased. This suggests that participants had a greater awareness of types of engineers after participating in camp activities, learning about engineers, and meeting engineers. Gathering the compilation of individual responses showed that multiple participants increased the numbers of engineering careers they could list. Furthermore, the most commonly named types of engineers were those that visited the camp to do a presentation.

These results suggest that participation in an engineering camp that focuses on career awareness can increase the number of possible career options in engineering that a student is able to consider. Studies of interest development and career choice in STEM suggest that exposure to available careers and engagement with the work of those careers can increase students' interest in pursuing those careers (Mohr-Schroeder et al., 2014; Wai et al., 2010; Yilmaz et al., 2010). Additionally, students who are exposed to more examples of engineers and their work at a young age may be more likely to view these options as possible future selves. This supports students in making the choices that might set them toward a career path in engineering because they envision that path as a possibility for themselves (Oyserman & Markus, 1990).

Attitudes toward engineering can encompass a variety of categories and therefore can be difficult to define. Lachapelle and Brennan (2018) discuss the development of attitudes toward engineering as the appraisal and judgement of "engineers, engineering as a profession, and learning experiences in engineering" (p. 222). The results from this study demonstrate that the engineering camp significantly increased students' perceptions of the value of engineering to society and aspirations toward engineering. First, this suggests that engaging in the work of engineers during a camp with projects focused on real-world scenarios can improve participants' attitudes about the role that engineers have in improving the world around us. Ing, Aschbacher, and Tsai (2014) demonstrated that seeing engineers making people's lives better improved interest in engineering careers, especially among females. Secondly, there was an increase in participants' aspirations toward engineering, which suggests that experience with engineers and engineering design may enhance participants' belief in their potential future as engineers. This supports the view that availability of role models and career exposure can open the door for young people to develop possible selves as engineers (Oyserman et al., 1995).

It should be noted that while the subscale for value of engineering to the individual was not significant, it was nearing significance (p = .055) and demonstrated a small effect size (r = .30). This is important because participants came in with generally strong positive attitudes toward engineering already. The average response was 3.9 out of 5.0 at the beginning of camp. While this alludes to the self-selected nature of the camp participants, the increase in multiple subscales indicates that the camp did have a positive effect on students' interests and attitudes toward engineering.

Limitations and Suggestions for Future Research

It is important to note the limitations associated with this study that impact the generalizability of the findings. First, camp participation was voluntary with either students self-selecting into the program and/or students' parents choosing to enroll them in the camp. Second, only a limited number of spots were available for participation each year due to resource limitations (e.g. facility space, personnel availability, cost of materials), and all participants hailed from the same midwestern community, limiting the geographic diversity of the sample. While we pooled data from two years of programming, the overall sample size was still relatively low.

Additionally, the data collected from the participants demonstrates the limited ideas that they were able to convey on paper instruments. While this data does provide worthwhile information for study, it would be beneficial to talk to students as well to gain a deeper understanding of their thinking. Future studies should include follow-up interviews with students to expand on this limited data, or interviews with parents to discuss the conversations that their children brought home during or after camp.

Despite the limitations, the study findings add to the knowledge base in engineering education and point to important areas for future research. The production of positive results indicates that the camp has many of the curricular characteristics that are beneficial for improving students' understanding, awareness, and attitudes. However, future camps will need to continue to improve on these results by continuing to emphasize career awareness and the work of an engineer. Because the camp did not significantly improve students' enjoyment of engineering, and the increase in understanding of the work of an engineer was small, future camps can also work to improve on each of these features. Finally, while the findings from this study indicate a positive result, there is little known about what causes students to move from their initial understandings and attitudes to their final positions. Future studies should focus on these incremental changes and how students interact with engineering design while they are working on hands-on projects.

Conclusion and Implications

This work adds to the literature on engineering education in two meaningful ways. First, the study illuminates how easily standard instruments can be influenced by the context of an intervention. In the current case, the WiE instrument was not able to discern changes in students' pre to post understanding of engineering due to the inclusion of the item "engineers teach children." When this item was removed from analysis, researchers were able to detect significant pre to post differences, indicating the need for researchers to carefully consider how instrument items might limit detection of gains under certain conditions. Furthermore, instruments may not always give an accurate depiction of a person's thinking or be able to detect changes, as evidenced by the lack of consistency in participant responses to similar questions on the WiE and EDPQ instruments.

Second, this work adds to the literature by providing evidence that an informal learning opportunity focused on engineering career awareness can enhance elementary-aged students' awareness of different disciplines of engineering and their aspirations towards engineering. The need for students who pursue an engineering career path is continuing to grow (World Economic Forum, 2017), and this need will only be met as students develop interest in engineering and believe that they have a possibility of becoming an engineer (possible self) in the future (Oyserman & Markus, 1990). While interest can be developed at any time, students who develop an interest in physical science or engineering careers by the 8th grade are 3.4 times more likely to pursue a career in those fields than those who are not interested at that time (Tai et al., 2006). Informal learning experiences such as camps have demonstrated promise in developing these interests in young students, and the engineering camp in this study adds to those results. The camp improved students' understanding of the work of an engineer, and most participants left camp with a greater awareness of the types of engineers and engineering careers that are available to them, possibly enhancing their ability to see a possible future self of engineer. Furthermore, participants' attitudes about the value of engineering and aspirations toward engineering became more positive through the camp. These findings suggest that a learning experience that incorporates hands-on activities that resemble the work of an engineer, a focus on types of engineering careers, and interaction with engineers can provide some of the pieces necessary to prepare a workforce that will meet the needs of our future society.

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General Proof Tasks Provide Opportunities for Varied Forms of Reasoning about the Domain of Mathematical Claims

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ABSTRACT

Proofs are attempts to conclusively demonstrate the validity of the claim for all cases indicated within its domain, which implies that proving should involve thoughtful consideration of the domain. This study analyzed the enactment of three general claim tasks, or tasks where the domain of the claim referred to an infinite number of cases, that were used during an introduction-to-proof teaching experiment with 10 ninth grade students. We analyzed the tasks in terms of the opportunities students experienced to engage in reasoning-and-proving and attend to the domain of the claims. The use of general claim tasks provided students with opportunities to engage in varied reasoning-and-proving activities, including forms not typically found in textbooks. Students' attention to the domain of the claims increased over the course of study as a result of the teacher-researcher's continued focus on this aspect of the tasks, although their attention did not always encompass all cases within the domain. By making the domain of mathematical claims a central focus, we emphasize its important role in the reasoning-and-proving opportunities afforded to students and contribute to an understanding of students' early interpretations of this aspect of proof tasks.

Keywords: reasoning and proving, instructional intervention, secondary mathematics, geometry

Introduction

The types of reasoning-and-proving tasks given to students impact their learning opportunities and shape the way in which they are able to reason about the mathematical content. With respect to reasoning-and-proving tasks used in high school geometry, students' opportunities are influenced not only by the validity of the claim, but also by the number of cases indicated within the domain of the claim (a single case, multiple but finitely many cases, or infinitely many cases) (Stylianides & Ball, 2008). For instance, claims involving a single case (e.g., prove a given triangle ABC is congruent to triangle DEF) can be essentially proven or certainly disproven by measuring the given sides and angles, whereas claims involving infinitely many cases (general claims) provide an intellectual necessity for a deductive approach[1]. In other words, since general claims cannot be proven using examples (Buchbinder & Zaslavsky, 2019), they are particularly well-suited to motivate the need for deductive reasoning (Otten, Gilbertson et al., 2014).

In addition to motivating deductive reasoning, general claim proof tasks in high school geometry courses can allow students to consider fundamental mathematical ideas. Because a proof of a general claim eliminates the possibility of counterexamples, it can result in the prover's ability to say with absolute certainty that a mathematical statement is true for all cases within the domain of the claim (e.g., Ellis et al., 2012; Fischbein, 1982). This is not to say that the production of a proof necessarily *convinces* a student the claim is always true (Rodd, 2000), just that it offers a level of certainty not afforded by examples. The ability to know with absolute certainty that a claim is always true is one way that mathematics (and physics) distinguishes itself from the biological sciences (Schoenfeld, 2000). Additionally, the use of both general and particular claims allows students to reflect on the number of cases encompassed within its domain, a worthwhile endeavor in and of itself that does not receive sufficient attention (Mason, 2019). Finally, the use of general claims reflects the broader practice of mathematicians who seek to pose and prove conjectures that encompass as many cases as possible.

Given the benefits of general claims, it is unsurprising that they are commonly used in studies focused on secondary students' understanding of proof and their ability to construct proofs (e.g., Buchbinder & Zaslavsky, 2019; Chazan, 1993; Healy & Hoyles, 2000; Knuth et al., 2009). This focus on general claims is not reflected within the reasoning-and-proving opportunities provided in textbooks. Otten, Gilbertson et al. (2014) analyzed U.S. high school geometry textbooks and found that student exercises involve particular claims much more than general claims. The discrepancy between the domain of claims used in proof tasks by researchers and those found in geometry textbooks highlights a need to better understand how the use of general claim tasks potentially impacts students' opportunities to (1) engage in the reasoning-and-proving process and (2) consider the domain of the claims being proven. Thus, this study examined a series of general claim tasks used during an introduction-to-proof teaching experiment with ten ninth grade students in the Midwest United States. Specifically, we examined the learning opportunities of proof-related tasks, as set up by the teacher-researcher, implemented with students and how students attended to the domain of the claims, evidenced in their written and verbal work. By making general claim proof tasks an explicit item of focus, we seek to promote greater understanding of the relationship between the domain of the mathematical claim and students' learning opportunities.

Defining Key Terminology

Reasoning-and-proving is used to refer broadly to all of the activity that goes into establishing the truth-value of a claim, from proposing a conjecture and investigating the validity of the claim, to constructing a proof or providing a justification that does not reach the level of a proof (non-proof rationale) (G. J. Stylianides, 2008). The term *proof* refers to "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" that uses acceptable justifications, valid modes of argumentation, and representations that are appropriate and understood by the classroom community (A. J. Stylianides, 2007, p. 291). Although Stylianides (2007) focused on the classroom community in an elementary setting, in the present study we interpreted the terms "valid", "acceptable", and "appropriate" according to both the classroom community and the broader

[1] There are specific mathematical claims that require deductive reasoning (e.g., prove that $2^{191} - 1$ is prime); however, these claims do not tend to be located in high school geometry textbooks. Within secondary education, claims occasionally fall into a separate category when they ask students to prove a claim for a relatively small number of cases (e.g., for numbers 1-20; see Knuth et al., 2009). For these tasks, students can reasonably check every single example (proof by exhaustion). That said, these claims tend to be numerical and are not typically used in high school geometry courses.

mathematics community because the context of secondary mathematics marked a shift toward formal proving. Additionally, we use the term *proof* as an adjective describing the tasks where students were expected to construct a proof and the term *argument* to refer to students' verbal or written work made in response to a proof task. Note that the term *argument* does not carry judgment about the quality of students' response or the extent it is aligned with Stylianides' (2007) and our definition of proof.

Recall that the *domain of mathematical claims* refers to the number of cases implicitly or explicitly referred to in the mathematical statement or theorem. Fischbein (1982) articulated the important role the domain of claims has in the proving process saying, "The level of generality of the theorem is then explicitly defined by the theorem itself and the proof refers exactly and clearly to that level of generality" (p. 15). In other words, theorems and statements to be proven indicate the domain of the claim (level of generality) and proofs demonstrating the validity of a given claim must clearly demonstrate it for all cases included within the domain of the claim. We continue with Otten, Gilbertson, and colleague's (2014) use of the terms *particular* and *general statements* to distinguish between proof tasks involving claims that reference a single case (particular statements) and those that encompass an entire, often infinite, set of cases (general statements). Geometric proof tasks that fall under the latter category typically use the quantifiers "all", "every", or "a" (i.e., "an arbitrary case") to indicate the domain of the claim.

Theoretical Perspective and Literature Review

Opportunities to Learn

Although there is a large body of literature focused on the teaching and learning of proof, few studies have specifically focused on students' opportunities to learn reasoning-and-proving with respect to the domain of the mathematical claims. Opportunity to learn originally referred to whether students, prior to being assessed, solved mathematical problems similar to those contained in the assessment (Husen, 1967, as cited in Floden, 2002). It is important, however, when determining students' opportunities to learn how to solve a certain type of problem, to disentangle the content topic of the problem and the specific formulation of the problem (Floden, 2002). In other words, it is one thing for students to be exposed to the mathematical ideas necessary to solve a problem; it is another thing for them to have practiced the exact type of problem presented in an assessment. As discussed above, proof tasks involving general claims may deal with mathematical content that is familiar to students, but their prior experiences may have been formulated with particular claims or situated within learning opportunities that did not draw attention to the domain of the claim.

The opportunities to learn framework can be particularly powerful when analyzing mathematics classrooms and instructional interactions given its demonstrated ability to connect teaching and learning (Hiebert & Grouws, 2007). The value of this perspective has emerged since the 1960s as opportunity to learn has come to be framed as more than topic coverage or problem-type familiarity; one can consider the topics together with the level of cognitive demand students' experience (Gamoran et al., 1997), the topics combined with the classroom learning environment (Tarr et al., 2013), or the interactions that occur in the classroom while topics are being taught (Jackson et al., 2013). In this study, we take the latter approach as we move beyond studying opportunities in textbooks (as summarized in the following sections) to a consideration of the opportunities that students have to think and discuss the reasoning-and-proving process *as* they work on tasks. In particular, we analyze the opportunities to engage in the reasoning-and-proving process within the tasks as launched by the teacher-researcher and in students' engagement in the tasks in order to allow for possible differences between the tasks' potential and realized opportunities (Stein et al., 1996).

Students' Attention during Mathematical Tasks

While it is certainly important for students to engage in the desired forms of reasoning-and-proving in order to increase their opportunity to learn this mathematical practice, doing so does not necessarily ensure that the intended learning will occur. Mason (2008) contended that "what teachers *can* do for learners, indeed perhaps the only thing they can actually do *for* learners, is to direct learners' attention [italics in original]" (p. 31). Ingram (2014) agreed, noting that students' attention can be influenced by features of the task and the interactions they have with teachers. Yet, even in instances where students and the teacher are collectively working on a single task, there is a potential for miscommunication to occur due to differences in where their attention is focused (Mason, 2008). When considering students' attention, one can focus on where that attention is directed but also the structure of attention. Structures of attention, according to Mason, include "holding wholes, discerning details, recognizing relationships, perceiving properties and reasoning on the basis of agreed properties" (2008, p. 35).

In order to understand the differences in the way that novices and experts attend to mathematical ideas, Mason and Davis (1988) coined the term "shifts in attention", which they defined as a moment, either sudden or gradual, "in which one becomes aware of what used to be attended to was only part of a larger whole, which is at once, more complex and more simple" (p. 488). During the proving process, students should begin shifting their attention away from specific details in examples or diagrams, towards a focus on generalizing through attention to mathematical relationships (Ellis, 2011). When interpreting mathematical statements being proven, students should recognize that the words *all*, *every*, and *any* indicate the impossibility of an exception that satisfies the criteria in the hypothesis but contradicts the conclusion (Harel & Sowder, 2007). Diagrams play an important role in many geometry proof tasks. Using diagrams when proving a general claim requires the ability to view the diagram as both an expression of generality (that is, a representation of all diagrams indicated within the domain of the claim) *and* as an object that can be manipulated (through rotations, adding notation, axillary lines, etc.) (Mason, 1989). Teachers can interpret diagrams as *figural concepts*, possessing both spatial properties and conceptual qualities (Fischbein, 1993) in part because they have been enculturated into the community of mathematics, wherein attending to the generality of mathematical claims is a central idea. In contrast, students who have not yet undergone this shift in attention may interpret the diagram only through the lens of its spatial properties or other features that are specific to the diagram drawn. Mason (1989) conjectured that "this is precisely where sophisticated mathematician-teachers, unaware of the momentary abstraction in themselves, miss the need to attend to the abstracting movement in their students" (p. 6). While it is important for teachers to attend to the ways that students are interpreting mathematical diagrams, shifting their attention towards the generality of claims is not something that a teacher can do or force onto students (Mason & Davis, 1988), nor is it something that can be achieved solely through calling attention to this aspect of mathematical claims (Mason, 2004). But students' opportunities to engage with general proof tasks may provide the context in which shifts in attention can occur.

Reasoning-and-Proving Opportunities in Textbooks

From the opportunities to learn perspective, students' thinking about reasoning-and-proving and the domain of mathematical claims is influenced by the opportunities embedded in curriculum materials. With respect to the introduction to proof chapters in U.S. Geometry textbooks, Otten, Males, and Gilbertson (2014) found that the student exercises primarily provided opportunities for them to investigate or pose conjectures and develop non-proof rationales, but few opportunities to construct a proof. Given that the introduction to proof chapter occurs early in Geometry textbooks, it makes sense that students' content knowledge might limit the number of proof tasks that are appropriate for the beginning of the school year. However, the limited opportunities to construct proofs suggests that students are developing their initial understanding of proof *without* actually engaging in the proving process. Looking at a random selection of the remainder of the Geometry textbooks, beyond the introductory chapters, Otten, Gilbertson et al. (2014) found that proof opportunities were prevalent, but they predominantly involved particular claims. The general claims that applied to infinite sets of geometric objects were typically presented in the textbook narrative, not the student exercises.

This focus on the domain of the claims is important because, although most reasoning-and-proving textbook studies (e.g., Fujita & Jones, 2014; Hanna, 1999; Miyakawa, 2012; Stylianides, G., 2009) have consistently examined the type of argument elicited (e.g., empirical, generic example, direct proof), it is by identifying the domain of the mathematical claims (i.e., general, particular, or general with particular instantiation) that we can consider whether those opportunities involved claims that *necessitate* a deductive proof. As Otten, Gilbertson et al. (2014) pointed out, deductive reasoning is powerful enough to establish the truth of both particular and general claims, but only deductive reasoning is able to establish the truth for general claims. Thus general claims necessitate deduction to a greater degree than do particular claims. Based on the findings that general claims were relatively rare in student exercises in U.S. Geometry textbooks, Otten, Gilbertson et al. (2014) called for future research analyzing the enactment of these opportunities in order to understand the role that the domain of mathematical claims might play with regard to students' experiences with proof.

Reasoning-and-Proving Opportunities in Classroom Settings

Research on proof instruction in secondary classrooms have primarily described whole-class conversations (e.g., Otten et al., 2017), providing only a snapshot into the reasoning-and-proving opportunities afforded to students in the classroom. Within this setting, teachers tend to spend a significant amount of time in their Geometry classes focusing on the details of proofs, such as whether each "step" in the proof contained a mathematically-correct justification and logically flowed from the previous statements (Martin & McCrone, 2003; Otten et al., 2017; Schoenfeld, 1988). Furthermore, traditional classrooms tend to operate based on specific norms around who is responsible for different aspects of the proving process (Herbst & Brach, 2006). For instance, students are rarely asked to prove their own conjectures (Boero et al., 2007); in instances where students are asked to conjecture, the teacher tends to confirm whether it is correct before students prove the claim (Herbst & Brach, 2006). In contrast to the teacher-driven reasoning-and-proving that occurred in the prior studies, Martin and colleagues (2005) described four classroom episodes where the teacher and students shared ownership in the reasoning-and-proving process. In these episodes, the teacher used revoicing and coaching in order to hold students accountable for contributing to the construction of the proofs. All of the whole-class conversation captured in the aforementioned studies focused on the task at hand (e.g., completing the proof) with little if any conversation that afforded students the opportunity to think broadly about reasoning-and-proving as a mathematical practice (Otten, Gilbertson et al., 2014).

Proof studies in secondary classroom settings have primarily occurred toward the middle or end of the school year; as a result, little is known about how students are first introduced to proof in traditional classrooms. One exception are the studies conducted by Cirillo (2011; 2014), who reported that six teachers introduced proof in their Geometry classrooms through a show-and-tell approach. During the teachers' proof demonstrations, Cirillo noted that the teachers did not explicitly unpack the many different components of proof, such as how they were using definitions to draw conclusions or what can and cannot be assumed from a diagram. In sum, there is still a need to better understand ways to introduce students to proof that utilizes a student-centered approach and develops students' understanding of proof through engaging in the reasoning-and-proving process, especially as those early opportunities relate to the domain of the claims being proved.

Students' Proving in Relation to the Domain of Mathematical Claims

One consistent pattern throughout proof research is the finding that a non-trivial percent of students construct empirical arguments for general proof tasks (Reid & Knipping, 2010; G. J. Stylianides et al., 2017). The construction of empirical arguments for general claims has been documented in studies of middle school students (Knuth et al., 2009), high schoolers (Healy & Hoyles, 2000; Lee, 2016; Senk, 1985), and undergraduate students (Harel & Sowder, 1998). Example use during the proving process is not inherently bad because students can productively use examples to gain insights into why a conjecture is true or uncover structural relationships (Aricha-Metzer & Zaslavsky, 2019). Nonetheless, students' use of examples as justification can reveal challenges in understanding that the goal is to construct an argument that applies for all cases, where no exceptions are possible (Harel & Sowder, 2007).

There are multiple possible explanations for why secondary students tend to produce empirical arguments when proving general claims. First, it is possible students recognize that empirical arguments do not prove general claims, but still write them because they lack the mathematical skills to be able to construct a more general argument (e.g., Bieda & Lepak, 2009; Healy & Hoyles, 2000; Reiss et al., 2001). However, this explanation does not account for instances when students use a few examples as justification in instances when proof by exhaustion would be a valid approach (Knuth et al., 2009). A second possibility is that some students misinterpret or do not recognize the domain of a mathematical claim due to a lack of explicit language indicating that the statement applied to an infinite number of cases (Mason, 2019). Or, it is possible that students are attending to the quantifiers indicating the domain of the claim but interpret them using a "real world" rather than mathematical definition (Pimm, 1987). These potential explanations speak to the importance of scholars not only attending to the empirical or deductive arguments that students produce, but also to their interpretation of the claim's domain.

While students can productively use diagrams as a planning tool or to capture their progress in a deductive argument (Cirillo & Hummer, 2021), others interact with diagrams in ways that suggest an interpretation of the diagram as a specific example (Herbst, 2004). Like Chazan (1993), Martin and colleagues (2005) found that students requested a proof for a second type of triangle even though the claim had been proven for a generic triangle, a request that suggests a lack of realization that the original proof demonstrated the claim was true for all triangles. In both instances, it is possible that the students were attending to generic and specific features of the diagram rather than interpreting it as a generic example. Infrequent opportunities to produce their own diagrams may also contribute to students' limited understanding of how to appropriately interpret a diagram. Although the norm of teachers or textbooks providing diagrams (Cirillo, 2018; Herbst & Brach, 2006) increases the consistency and accuracy of the diagrams students use, it limits their opportunities to reason about what the diagram represents or about the generality indicated within the proof claim (Komatsu et al., 2017).

Research Questions

Collectively, prior research on students' understanding of proof and the ability to construct proofs highlights a need for changes to the ways that proofs are taught in the classroom, particularly in order to fulfill the recommendations that reasoning-and-proving should be a central part of K–12 instruction (Ministry of Education, Science and Technology, 2011; National Council of Teachers of Mathematics, 2009; National Governors Association & Council of Chief State School Officers, 2010). This study extends the textbook analysis of Otten, Gilbertson et al. (2014) by analyzing the enactment of three general claim tasks in terms of the reasoning-and-proving opportunities they afforded, and students' attention to the domain of mathematical claims. The research questions that guided the analyses were as follows:

RQ1. What opportunities for reasoning-and-proving were present in general claim tasks set up by the teacher-researcher and implemented with students during an introduction to proof unit?

RQ2. How, if at all, did students attend to the domain of the claims and what, if any, shifts in attention occurred over the course of a task enactment?

Both questions were addressed through an analysis of students' oral and written work on the tasks, with the data and analytic processes described in detail in the next section.

Method

Participants and Data Collection

Ten students participated in this study—seven females (Amanda, Arin, Heather, Lauren, Lexi, Megan, and Sadie) and three males (Brian, Clay, and Wilson; all pseudonyms). They were the only students enrolled in an accelerated ninth grade mathematics course at a rural, public school in the Midwest United States. The accelerated program covered Algebra 1 and 2 content in ninth grade; subsequently, the study provided the students first formal high school Geometry instruction. All sessions were held during the school day but outside of their regular mathematics class. Students received a graphing calculator for their participation.

The exploratory teaching experiment (Steffe & Thompson, 2000) consisted of 14 sessions, held twice a week with each session lasting between 28 and 38 minutes. All sessions were taught by the first author, who is identified as the teacher-researcher (TR) in this article. The use of a researcher as the teacher is consistent with teaching experiment methodology (e.g., Cobb & Steffe, 1983; Steffe & Thompson, 2000) and should not be confused with self-study methodology, wherein the researcher studies their own teaching in classrooms where they are the main instructor. Every session was video and audio recorded to ensure that students' gestures, manipulation of physical objects, and voices during small-group discussions could be reviewed, with one audio and video recorder placed near each group. Additionally, all written work and students' responses to journal prompts were collected during the sessions. For this study, audio/video recordings served as the primary data source; students' written work and journal reflections were referenced as needed in order to provide a more complete picture of what occurred during the sessions.

Teaching Experiment Design and Rationale

Exploratory teaching experiment methodology is used to study students' ways of understanding and operating with particular content in instances when testing the researchers' hypotheses for learning may not be appropriate (Steffe & Thompson, 2000). In particular, this methodology was selected in order to better understand students' ways of understanding proof while engaging in tasks that are not commonly found in traditional classrooms. The primary goal of the present study was to develop students' understanding of the purpose of proof through their engagement in tasks that emphasized the proving process as a means of a) developing certainty that the given statement is *always* true and b) understanding *why* it is always true (de Villiers, 1990; Hanna & Jahnke, 1996). Specifically, we hypothesized that the explanatory feature of proofs could help students transition away from empirical arguments since examples on their own do not tend to explain why a statement is true. We chose to only use tasks involving general claims based on the hypothesis that they could facilitate student understanding that a proof must contain justifications that encompassed all objects within the claim's domain, particularly when accompanied by conversations where the domain was an explicit object of focus.

The TR structured the instruction so that students developed their understanding of proof as they engaged in various reasoning-and-proving activities. The goal to engage students in authentic reasoning-and-proving has been used in a variety of intervention-based studies (e.g., G. J. Stylianides et al., 2017). For example, the present study's use of general claims and having students prove their own conjectures was successfully used in a study with eighth graders (Boero et al., 1996). Finally, we incorporated statements *about* reasoning-and-proving (Otten, Gilbertson et al., 2014) into whole-class discussions and through the use of reflection prompts in order to focus students' attention on specific aspects of proofs. Although hypotheses for learning were developed to guide the instruction, additional iterations of the teaching experiment would be needed in order to test and revise the instruction so that students' ways of understanding aligned with the researcher's hypotheses (or the hypotheses could be revised upon further iterations). During the sessions, the TR did not focus on the form of proofs, but instead allowed students to write arguments in a way that made sense to them.

The primary tasks used in this study were developed before the start of the experiment based on the hypotheses described above. On the other hand, the time spent on each task and select sub-tasks were devised during the study in response to where the TR interpreted students to be in their current understanding. For example, the original tessellation task, "do all quadrilaterals tessellate?" was pre-planned, but the follow-up task, "do all regular polygons tessellate?" was added mid-experiment in an attempt to focus students' attention on both the sides and angles of polygons. We describe the three focal tasks in the following sections; see Appendix A for a description and rationale for an overview of the entire instructional sequence.

Overview of the Tessellation Tasks

After reviewing the definition of quadrilaterals and introducing tessellations, the TR launched the tessellation task by posing the question, "Do all quadrilaterals tessellate?" Students were given six sets of different convex quadrilaterals to aid in their investigation. At the end of the session, the TR asked students to journal how confident they were that quadrilaterals always tessellate and to describe how they would explain their answer to a friend. In Session 2, students were asked to write a set of "step-by-step" directions for how to tessellate *any* quadrilateral. Each group was given some of the quadrilaterals from Session 1 as well as two concave and one convex quadrilaterals to use during the task. In Session 3, the TR introduced the next subtask by asking students if they knew of other polygons that they thought would always tessellate. After eliciting their ideas, the TR introduced regular polygons and referenced familiar examples. Next, the TR posed the question, "Do all regular polygons tessellate?" Regular polygons were selected because they fit within students' directions for tessellating quadrilaterals, despite only some tessellating. In order to investigate this question, the TR first provided all groups with a set of regular hexagons and then passed out regular pentagons, septagons, and octagons (one per group) to "speed up" the process. The reflection prompts provided in Session 3 (see Appendix A) were used to encourage connections across the tasks and motivate a need to understand *why* quadrilaterals always tessellate. The term "counterexample" and the idea that only one counterexample was needed to disprove a general claim was introduced to students towards the end of Session 3. The TR concluded the tessellation task in Session 4 by summarizing the key ideas from the first three sessions and then explaining why quadrilaterals and regular hexagons, but not regular septagons or octagons, tessellate.

Overview of the Constructing Quadrilateral Diagrams Task

Prior to launching the diagrams task, the TR briefly introduced students to conditional statements and demonstrated how they are used in the proving process by talking through an informal proof of the conditional statement, "If a quadrilateral has 360°, then it will tessellate."[2] Afterwards, students worked in three small groups to draw diagrams for statements 1–3 during Session 7 and statements 4–6 during Session 8 (Table 1). Since they had not yet taken high school Geometry, the theorems were rephrased to exclude potentially unfamiliar terminology such as "congruent," "consecutive angles," and "supplementary." At the end of each session, the whole class discussed specific features of the diagrams, including the different notation methods they had used.

Table 1

Statements Used in the Constructing Diagrams for Quadrilateral Theorems Task

1.	If the polygon is a rectangle, then the diagonals have the same length.
2.	If a quadrilateral is a parallelogram, then the measures of the angles on the same side of the shape add to 180 degrees.
3.	If a quadrilateral is an isosceles trapezoid, then the diagonals have the same length.
4.	If two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus.
5.	If the diagonals of a parallelogram form a 90-degree angle, then the parallelogram is a rhombus.
6.	If one angle of a parallelogram is a right angle, then the parallelogram is a rectangle.

Overview of the Proving Similar Polygon Conjectures Task

The TR introduced the similar polygons task by asking students to pose conjectures of specific polygons that they thought might be similar (i.e., "all ____ are similar"). During the launch, they discussed the teacher-posed conjecture "all polygons are similar" to make sure that students understood the conjecture and remembered how to use a counterexample to disprove a conjecture. Students posed four conjectures that involved classes of polygons that were always similar to one another: squares, equilateral triangles, right triangles, and rhombuses. After constructing their argument for the conjecture "all squares are similar", small groups exchanged papers and provided feedback to their peers. The TR posed the following questions to focus students' feedback: "Is it convincing? Does it convince you that no matter what two squares I draw, they're going to be similar? And is there anything someone could say to poke a hole in the argument?" Next, each group revised their own argument in response to the two sets of peer feedback they received. In Session 12, the TR led the entire class through the proof of the conjecture, "all squares are similar", which built on elements that were in students' arguments from the previous session. Afterwards, students investigated the classes' conjectures for right triangles and equilateral triangles and then constructed an argument demonstrating that the conjecture was either true or false. Students investigated the final conjecture, "all rhombuses are similar", during the final interview and then constructed an argument either proving or disproving it, depending on their belief in the conjecture's validity.

Analytic Process

In addition to rooting our study in the literature previously described and explaining the relationship between the researchers and study participants, we now articulate our process in the data

[2] This claim was stated by one of the students in an earlier session. We chose to use their phrasing in order to connect to the student's earlier words instead of starting with a claim that was more mathematically precise. During the class discussion, the TR clarified that the hypothesis referred to the angles of a quadrilateral.

reduction and analysis process in order to make claims using qualitative research methodology (Noral & Talbert, 2011). We restricted our analysis of the data to sessions involving geometric tasks. There were five broad geometric tasks: the tessellation tasks (Sessions 1-4; 103 minutes), constructing diagrams task (Sessions 7-8; 68 minutes), constructing a definition for similar polygon conjectures task (Sessions 9–10; 62 minutes), proving similar polygon conjectures task (Sessions 11–12; 157 minutes[3]), and proving the exterior angle theorem task (Session 13; 34 minutes). The unit of analysis was a response (written and/or verbal) to one question or prompt (subtask) within the identified tasks. Specifically, units of analysis spanned the time between when student(s) started and completed each subtask in whatever grouping configuration they were placed. Most subtasks were completed in three small groups; however, six subtasks (four reflection and two math prompts) were completed individually. We excluded whole-class discussions in instances when they only reiterated students' small group work so as not to double analyze reasoning-and-proving activity. Collectively, there were 119 units to be analyzed. In this article, we present findings for the tessellation, constructing diagrams, and proving similar polygon conjectures tasks since they best illustrate the range of reasoning-and-proving that occurred.

To answer RQ1, we analyzed the session data using the qualitative research software MAXQDA to determine the opportunities students had to engage in reasoning-and-proving based on the launch and implementation of each task. In order to make comparisons between the types of reasoning-and-proving opportunities found in regular Geometry textbooks and the instruction used in this study, we adapted the expected student activity portion of Otten, Gilbertson et al.'s (2014) analytic framework, which was a modified version of Thompson and colleagues (2012)'s framework (see Table 2).

Table 2

Reasoning-and-Proving Student Activity Codes

Related to Mathematical Claims	Related to Mathematical Arguments	Emergent Codes
• Make a conjecture, refine a statement or conjecture, or draw a conclusion • Fill in the blanks of a conjecture • Investigate a conjecture or statement	• Construct a proof • Develop a rationale or non-proof argument • Evaluate an argument/proof • Find a counterexample	• Make sense of a mathematical claim • Construct a diagram • Revise an argument/proof

Note. The codes in the first two columns are from the framework described by Otten, Gilbertson et al. (2014).

We adapted Otten, Gilberston et al.'s (2014) framework such that it applied to both the anticipated reasoning-and-proving activity and the reasoning-and-proving students actively engaged in during the sessions. For example, in the exchange below during the proving similar polygons task, both students' comments were coded as *develop a rationale or non-proof argument*. Additionally, the entire exchange was included within a broader *evaluate an argument/proof* code to capture the broader reasoning-and-proving activity being completed.

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[3] Students individually completed the final prompt of this task during the final interview; this accounted for 93 of the 157 minutes.

Wilson: I don't think [the angles] should be [labeled] A, B, C, D, I think it should be A, A, A, A cause they're all the same angle

Megan: and then they need those [notation] on the edges [sides] of the square to show that it's the same length cause that's what makes it a square.

Develop a rationale or non-proof argument was used in instances when students provided a justification for a single statement ("because…") and in instances when students were asked to "explain" or provide a justification for "why" a claim is true. In contrast, *construct a proof* was used in instances when the task directly asked students to prove a mathematical claim. Note that the presence of this code did not guarantee that the resulting product contained all of the required elements to be considered a proof. During the coding process, we identified additional instances of reasoning-and-proving that occurred in the sessions but were not captured by Otten, Gilbertson et al.'s (2014) codes. This resulted in three additional codes: *make sense of a mathematical claim*, *construct a diagram for a mathematical statement*[4], and *revise an argument/proof. Make sense of a mathematical claim* was used in instances when students talked about a claim without trying to determine whether it was valid (the latter would be coded *investigate a conjecture or statement*). For example, Arin's second statement below, which occurred during the constructing diagrams for quadrilateral theorems task, was coded *make sense of a mathematical claim* because she was not trying to actively determine if the claim was true.

Arin: *(reading)* "If a quadrilateral is a parallelogram" is that the like one that's straight lines and then it like *(makes a slanted line gesture with her hands)*

Sadie: Yeah

Arin: Okay. *(reading)* "then the measures of the angles on the same side of the shape add to 180." Yeah, because one angle is going to be bigger than the other.

Instances when students discussed mathematical vocabulary, such as Arin's first sentence, were not coded as *make sense of a mathematical claim* because it was activity outside of the reasoning-and-proving process. Finally, the code *revise an argument* referred to instances when students revised their draft argument in response to peer feedback. To increase trustworthiness (Lincoln & Guba, 1985), the authors had continual calibration conversations with one another and also produced preliminary analytic memos that were vetted by an outside observer.

To answer RQ2, we first analyzed students' discussions during each task in order to assess how, if at all, students were attending to the domain of the claims. Although it is not possible to ascertain what students were internally attending to at a particular moment in time, we could look for evidence of shifts in attention through what they said or did in their conversations with peers (Barwell, 2002, as cited in Ingram, 2014). Examples of students demonstrating attention to the domain of the claim include a student responding to a peer's assertion by saying, "no that's not true for all of them". Next, we analyzed students' written work for evidence of attention to the domain of the claims. Specifically, we determined whether students' justifications, constructed diagrams, and notation methods encompassed all cases within the claim's domain. Although we coded students' written work as indicating attention to the domain of the claim, or a lack thereof, we recognize the possibility that a student could construct a general argument or notate their diagram with variables for reasons other than their understanding of the claim's domain. Additionally, it is possible that a student could

[4] Constructing a diagram could have been coded using the "modify or revise a mathematical statement" code since students were adding a diagram to accompany the provided statements. However, a new code was added to emphasize the fact that textbooks and teachers rarely, if ever, hold students responsible for producing a diagram for a proof task (Cirillo, 2018; Herbst & Brach, 2006).

recognize the domain of the claim but produce an argument that only refers to a finite number of cases.

After coding for individual instances of attention to the domain of claims, we then looked across the coded data for evidence of shifts in attention (Mason & Davis, 1988). For example, if a group of students initially labeled their diagram with specific angle or side measurements in one instance, but later labeled them with variables, this would indicate a shift in attention to the fact that the mathematical claim being represented by the diagram refers to an infinite class of quadrilaterals. All instances where the students' attention to the domain of the claim was unclear were discussed with an outside observer; in these instances, we include possible alternate interpretations in the results.

Findings

We describe students' engagement in three general claim tasks in terms of the reasoning-and-proving opportunities that surfaced during the tasks. We then share students' interpretation of and attention to the domain of each mathematical claim. The findings are structured by task in order to (1) highlight the range of reasoning-and-proving afforded within a single task, and (2) acknowledge that the mathematical content of the task (Dawkins & Karunakaran, 2016), other features of the task, and its location in the instructional sequence may have influenced students' attention to the domain of the claims. Across the three tasks, students engaged in all of the reasoning-and-proving opportunities set forth in the launch of the tasks as well as additional, unplanned reasoning-and-proving that arose during small-group and whole-class conversations. Although students demonstrated limited attention to the domain of the claims at the beginning of the tessellation task and constructing diagrams task, there were small shifts in attention by the end of both tasks. In contrast, during the proving similar polygon conjectures task, students attended to the domain of the claims throughout their conversations and through their written justifications. However, some students attended to the domain of the claims in a way that did not encompass all possible cases during the rhombus portion of the similar polygons task.

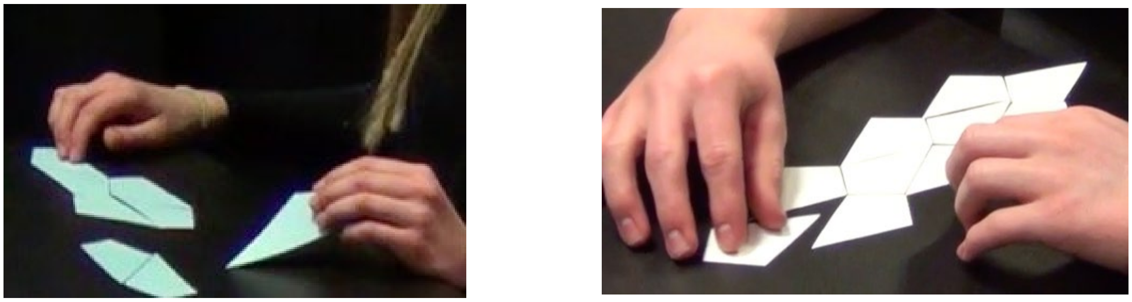
Tessellation Tasks

Varied Reasoning-and-Proving Opportunities

The tessellation tasks, as launched by the teacher-researcher (TR), provided students with opportunities to investigate the validity of mathematical statements, construct a counterexample, and develop non-proof rationales for their assertions. Specifically, Session 1's subtask ("do all quadrilaterals tessellate?") resulted in non-proof rationales as students investigated how to tessellate different quadrilaterals in their small groups (Figure 1) and again as they individually summarized their responses in their notebooks. In Session 2, the subtask to create "how to" directions for tessellating *any* quadrilateral did not explicitly provide opportunities for students to engage in reasoning-and-proving, but did encourage greater attention to the domain of the mathematical claims.

Figure 1

Two Students Tessellating Different Irregular Convex Quadrilaterals in Session 1.



Session 3 involved two subtasks; the first, "do all regular polygons tessellate?", allowed students to investigate a conjecture and find a counterexample. The second subtask ("Do you still think that all quadrilaterals tessellate? If no, explain why. If yes, is there something special about quadrilaterals that make it so that they will always tessellate?") provided opportunities for non-proof rationales. In addition to engaging in all of the intended reasoning-and-proving activity, students also made conjectures, posed counterexamples in response to a peer's conjecture, and refined a peer's conjecture. The additional reasoning-and-proving activities occurred in Session 3 while students discussed as a whole class the possible characteristics of polygons that tessellate.

While investigating the validity of the claim ("do all quadrilaterals tessellate?"), students' non-proof rationales in their written reflections at the end of Session 1 referenced the different cases that had successfully tessellated and an assumption that the pattern would continue to hold true for other cases. For example, Sadie wrote, "I'm very confident that all quadrilaterals tessellate. Since we tested out many different shapes and they all worked, it helps prove my point. I would convince [a friend] by showing them how I found out that they all fit." Similarly, Amanda wrote, "all quadrilaterals tessellate because if you match up one side of the quadrilateral, the other sides will have to match up too." Across students' written work at the end of Session 1, nine students included non-proof rationales to justify why they thought all quadrilaterals tessellate.

The final subtask around why all quadrilaterals, but not all regular polygons, tessellate produced the most varied opportunities for reasoning-and-proving, including activities that were not requested in the original prompt. After writing down their justifications, students discussed in small groups and then as a whole class possible reasons why only some polygons tessellate. The dialogue below occurred during the whole-class conversation.

Amanda: I said that maybe after a shape gets like, like after they have four sides, like five and on, then maybe the angles become too wide, because they have too many sides

TR: Okay. What do y'all think about that?

Wilson: Well, hexagons work, but…

TR: So hexagons work… Lexi, can you speak up a little bit?

Lexi: Okay, well I said maybe. *(Arin quietly interrupts her)*

TR: Go ahead [Lexi] and say what you were thinking.

Lexi: Okay, well I said maybe like after four sides the sides have to be even with the amount, cause five didn't work.

TR: Five didn't work, yeah; so that would be a reason for that…but what, hold on, will you talk a little bit louder?

Arin: The octagon didn't work.

In this exchange, Amanda and Lexi posed conjectures that described features of polygons they thought would tessellate (or not) and Wilson and Arin responded to each claim with a counterexample the class had previously investigated. The fact that hexagons tessellate was a counterexample to Amanda's idea that polygons with more than four sides could not tessellate, and octagons failing to tessellate was a counterexample to Lexi's idea that even-sided polygons might tessellate. The discussion is notable given that students had not yet been formally introduced to the use of counterexamples in the proving process or the idea of revising a claim. Instead, the additional unplanned reasoning-and-proving activity surfaced as students discussed two general claims (do all quadrilaterals tessellate? And do all regular polygons tessellate?) that were similar in structure but differed in validity. Through the use of two such general claim tasks, students were not only able to investigate the validity of the claims and provide non-proof rationales, but were also able to pose conjectures and counterexamples by looking across the two tasks.

Increased Attention to the Domain of the Claim

Throughout Session 1, students' justifications relied on a lack of counterexamples rather than the identification of specific features of all quadrilaterals that result in them tessellating. Thus, there was little explicit attention on the domain of the claim. Students' attention to the domain of the claim increased during Session 2 as they developed a series of "how to" directions for tessellating any quadrilateral. For example, Megan and Arin's written directions stated: "1st we put opposite angles together. 2nd we repeated the first step as well as flipped and mirroring the shapes from the original two shapes. Same side length, different angles". These directions represent a shift in attention from haphazardly moving copies of a quadrilateral around until they "fit" to purposefully placing the quadrilaterals together by focusing on the sides and angles. When describing how to tessellate any quadrilateral, Megan and Arin referred to generic sides and angles and did not mention more specific features that some, but not all, quadrilaterals contain (such as 90 angles or congruent sides). The reference to generic features of quadrilaterals may have been a result of the task prompt to create step-by-step directions for how to tessellate *any* quadrilateral rather than a change in how students were interpreting the different provided examples, but regardless, the shift in attention was noteworthy.

Small-group conversations in Session 3 revealed students' varying attention to the domain of the claim. For example, Clay suggested they put two quadrilaterals together so that they make a nicer shape such as a rectangle or square, and Heather responded by saying, "that's just for this shape, it's not for all of them. Like different quadrilaterals make different shapes, not just a square." Although Clay appeared to be focusing on features of certain quadrilaterals, Heather's response suggests that she was considering multiple quadrilaterals when thinking about how to place the two copies together to form a tessellation. In the whole-class discussion around features of polygons that determine whether they will tessellate in Session 3, both Amanda and Lexi's justifications referenced general features of polygons (the number of sides and angles) rather than specific characteristics. Students' increased use of statements that applied to multiple if not all quadrilaterals in Sessions 2 and 3 suggest at least some attention to the domain of the claims. Given the explicit emphasis on *all* or *any* when launching the subtasks, it is possible that students' use of these words in their conversations reflected the instructional focus rather than how they were mentally thinking about the claims (Mason, 2004). Nonetheless, students' written work and conversations revealed moments where at least some students appeared to be considering multiple, if not all possible cases.

Constructing Diagrams for Quadrilateral Theorems Task

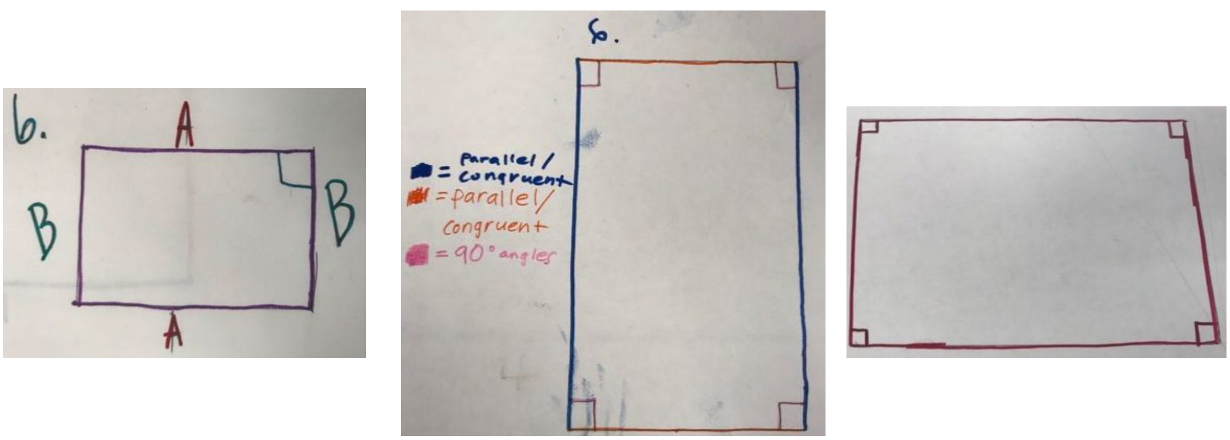
Limited Reasoning-and-Proving Opportunities

The constructing diagrams task, as launched by the TR, provided students with the opportunity to construct a diagram for six quadrilateral theorems (see Table 1 for task directions and Figures 2–4 for examples of student-constructed diagrams). In addition to engaging in the intended reasoning-and-proving activities, students also informally drew conclusions and made sense of the mathematical claims. For example, the three small groups' diagrams for the theorem, "if one angle of a parallelogram is a right angle, then the parallelogram is a rectangle" are shown in Figure 2.

As students constructed the diagrams, some began informally drawing conclusions from the hypotheses by applying their prior knowledge of quadrilaterals. While constructing the left diagram in Figure 2, Wilson argued that all of the angles had to be 90 degrees based on the given information. "This one has to be 90 degrees since...they're all 90 degrees, yeah. Because these (adjacent angles) have to add to 180 and if one of them is 90 degrees, the other has to be 90 degrees." Even though the task did not ask students to construct proofs for the given theorems, it allowed the opportunity for students to begin informally reasoning about the theorem and verbally begin to draft a rough outline for a mathematical argument.

Figure 2

The Three Small Groups' Diagrams for the Theorem, "If One Angle of a Parallelogram is a Right Angle, then the Parallelogram is a Rectangle."



Note. The legend in the middle diagram reads, blue (vertical sides) = "parallel/congruent"; orange (horizontal sides) = "parallel/congruent"; pink (angles) = 90° angles.

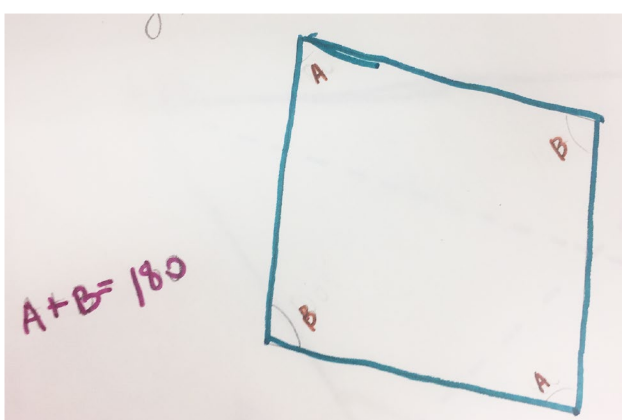
In addition to opportunities to construct diagrams and informally draw conclusions, the task also provided opportunities for students to make sense of each claim. This was especially true for the last three theorems in the task (Table 1), since the theorems referenced a different quadrilateral in the hypothesis and the conclusion. When constructing a diagram for the theorem, "if one angle of a parallelogram is a right angle, then the parallelogram is a rectangle", Arin, Sadie, and Brian initially constructed the figure by drawing a right angle and then a slanted line "because parallelograms have slant." As a result of only attending to the information in the hypotheses, their diagram resulted in a right trapezoid rather than a rectangle. Through a discussion with the TR, the students were able to connect their understanding of the definition of a parallelogram to recognize that they could construct a rectangle to satisfy both the theorem's hypotheses and conclusion. While the constructing diagrams task afforded more limited opportunities for reasoning-and-proving, the use of general claims for the task required students to make sense of the claims and afforded opportunities to informally draw conclusions from the hypotheses. Although often overlooked, reasoning about the claim itself can lay an important foundation to support students in the proof construction process (Cirillo & Herbst, 2011).

Attention to the Domains of the Claims in Relation to Diagrams

Since students were asked to use their own notation methods, this aspect of the diagrams provided insights into how they were attending to the domain of the claims. Of the nine diagrams students constructed in Session 7, four of them suggested that students paid limited attention to the domain of the claims either in the type of quadrilateral they drew, or in their selected notation method. In contrast, only one of the nine diagrams constructed in Session 8 contained specific notation that did not encompass all objects within its domain. For instance, Lauren, Megan, and Wilson first labeled the angles of their parallelogram 100º and 80º when constructing a diagram for the statement, "If a quadrilateral is a parallelogram, then the measures of the angles on the same side of the shape add to 180 degrees." When asked if those were the *only* angle measurements for a parallelogram, Wilson replied, "I don't know, those probably don't even, they add to 180 I know that, but those probably aren't the exact measurements you know." Even though this group originally labeled their angles with specific measurements, Wilson's justification suggests that he had chosen the angle measurements arbitrarily and had not based them on the actual measurements in their diagram. As a result of the TR's question, Megan proposed changing the labels to "A, A, B, B" and Wilson suggested adding the equation "A + B = 180" (Figure 3). Note that their small-group conversation did not reveal any evidence that they intended their diagram to be a rhombus instead of a parallelogram (Wilson: "just draw a parallelogram…just one that looks nice."). At the end of Session 7, both Megan and Wilson stated during the whole-class conversation that they preferred the use of variables to notate the sides of the rectangle because variables were "more generic."

Figure 3

Lauren, Megan, and Wilson's Revised Diagram for the Theorem, "If a Quadrilateral is a Parallelogram, then the Measures of the Angles on the Same Side of the Shape Add to 180 Degrees."

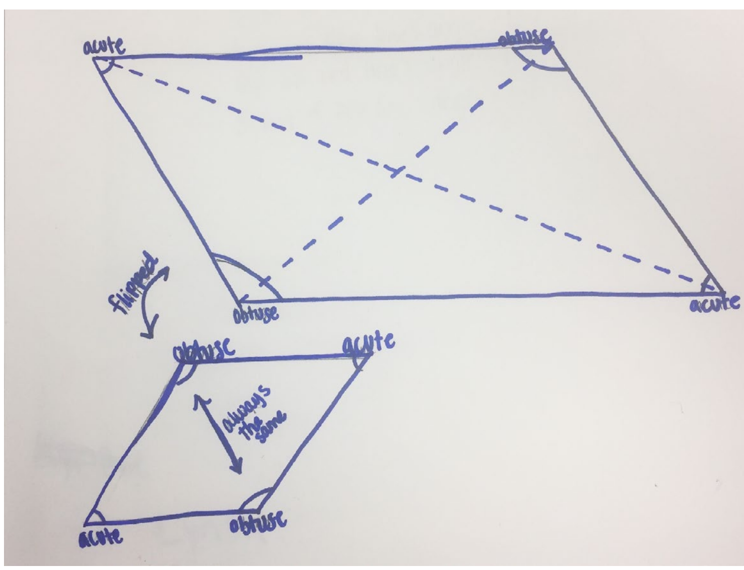


Arin, Brian, and Sadie's diagram for the same statement consisted of a general parallelogram; however, their decision to label the angles as "acute" and "obtuse", omitting right angles, made the notation less general than the prior group's use of variables (Figure 4). After being asked if the specific angles of parallelograms would always be acute and obtuse as they had labeled them, Arin replied "no, it could change. Like if the lines were drawn [in the opposite direction], then this [acute] angle would be obtuse." In response to this exchange, Sadie drew a second, smaller diagram containing different angle labels (Figure 4). Although the constructed parallelogram is generic, their choice to label the angles as obtuse or acute could result in a corresponding mathematical argument that makes assumptions about the angles that are not true for all cases (e.g., the upper left angle is acute). Sadie's decision highlights one of the challenges of constructing diagrams for general statements: namely, that it is impossible to construct a diagram that has the features of *all* possible shapes.

At the beginning of the constructing diagrams task, there was limited evidence that students were attending to the domain of the claims. However, their attention to the domain of the claim increased after the TR questioned groups whether their diagram applied to all possible shapes. At the end of Session 8, students appeared to have a greater awareness of the different ways that diagrams could be drawn to represent general claims. In a written reflection, Amanda explained that "it is okay that the diagrams didn't look the same because not all shapes may look the same, but they still fit the requirements to be that shape." Amanda and others recognized that diagrams can vary in their appearance so long as they contain all of the features specific to the shape mentioned in the general claim. Students' use of their own notation methods when labeling diagrams for general claims allowed for greater insight into their attention to the generality of the claims and highlighted the challenge students can face in constructing a single diagram to represent a class of objects.

Figure 4

Arin, Sadie, and Brian's Diagram for the Theorem, "If a Quadrilateral is a Parallelogram, then the Measures of the Angles on the Same Side of the Shape Add to 180 Degrees."



Proving Similar Polygon Conjectures Task

Varied Opportunities for Reasoning-and-Proving

The proving similar polygon conjectures task, as set up by the TR in Session 11, provided opportunities for students to pose conjectures about certain polygons that might be similar (e.g., "all squares are similar"), investigate the validity of their conjectures, either construct a proof or find a counterexample, construct a diagram to accompany their argument, evaluate their peers' arguments, and revise their argument based on peer feedback. In addition to engaging in all of the reasoning-and-proving activities set forth in the task, some students also constructed non-proof rationales and posed a revised conjecture while evaluating their peers' arguments.

To illustrate the different reasoning-and-proving opportunities embedded within this task, we describe the original argument constructed by Group 1 (Arin, Brian, and Sadie), the feedback given to Group 1 by Group 2 (Megan, Wilson, and Lauren) and Group 3 (Clay, Amanda, Heather, and Lexi), and then the revisions Group 1 made to their argument in response to the provided feedback (see Appendix B for final work). Group 1 worked on the claim that all squares are similar. Group 1's original argument included a diagram consisting of two different-sized squares with no notation on the sides and the angles labeled with the variables A, B, C, and D. Their written argument stated: "If all of the angles on a square are 90 degree angles, then they are the same. If all sides have the same measurements, then they will be proportional." Note that each sentence in their argument addressed one component of the definition for similar polygons. However, they did not justify how they knew the sides would be proportional or explicitly mention the definition of similar polygons or squares.

When evaluating the argument written by Group 1, Megan, Wilson, and Lauren focused on the way the group had chosen to label their diagram.

> *Megan*: I think that they should put…like, if they're going to do like letters then there should be ones on the sides too because that's, like, what makes it a square.
>
> *Wilson*: I think they should all be (unintelligible), I don't think they should be A, B, C, D, I think it should be A, A, A, A cause they're all the same angle.

Notice that both Megan and Wilson provided a non-proof rationale (e.g., "they're all the same angle") to justify their proposed revisions. After a discussion with the TR, they concluded that the angles could be labeled with 90° instead of a variable. This groups' final feedback included a revised diagram with the angles all labeled 90° along with the statement, "If the angles are the same, the side measurements will be proportional." Their feedback assumed a relationship between congruent angles and proportional sides, however it had not yet been discussed how to demonstrate that the sides of squares were proportional for all cases.

Clay, Amanda, Heather, and Lexi (Group 3) provided feedback by underlining Group 1's use of the words "same" and "proportional" at the end of each sentence and then writing, "We're not trying to prove that they are proportional, but that they are similar." This critique suggests that they may not have recognized that each sentence in the original argument referred to one of the components of the definition for similar polygons. Nonetheless, it highlighted the need for Group 1 to use more precise language in their original argument or to more clearly lay out the broad goals of their argument. During the revision process, Group 1 tweaked the first sentence to clarify that the *angles* are the same in response to Group 3's feedback. They also revised their angle notation in the diagram and added labels to the sides of the two squares in response to Group 2's feedback.

In the proving similar polygons task, students had the opportunity to engage in a variety of reasoning-and-proving activities as they developed and honed their understanding of proof. Although none of the groups produced arguments that contained all of the elements and formatting of a traditional proof (which, after all, was not expected), their work on the task was notable given that this was their first formal experience constructing a proof. Students were actively involved in the decision-making during this task, evaluated each other's feedback, and decided whether they wanted to incorporate it into their revised argument. The directions within this task not only allowed students to experience varied reasoning-and-proving in a connected, authentic way, but also allowed students to enact their role as a member of the proving community through revising their argument based on peer feedback.

Attending to the Domain of the Claim in Varied and Complex Ways

When proving that all squares are similar to one another, all three groups wrote arguments containing justifications that reflected an attention to the domain of the claim. Additionally, two of the three groups constructed a single diagram to accompany their argument that used variables to label the sides and, in one group, the angles as well. After completing their initial written argument, the remaining group (Heather, Amanda, Lexi, and Clay) chose to "draw another square", which they labeled with specific side lengths, "to show that they all work". It is not clear from this group's discussion whether they saw the specific diagrams as part of their core mathematical argument or as further evidence to convince someone the claim was true. Finally, all three groups also demonstrated understanding that a single counterexample proved a general claim was false for the right triangle conjecture.

Students' arguments for the (false) claim that all rhombuses are similar revealed more variation in attention to the domain of the claims, in part because they were completed individually during the final interview instead of within their small groups. Six of the ten students demonstrated understanding of the domain of the claim, "all rhombuses are similar", through their use of a single counterexample or class of counterexamples to prove the claim was false. Specifically, they argued the claim was false by giving a specific counterexample (Amanda, Lauren, and Sadie), mentioning that the angles of a rhombus "have no set rule" (Megan), or providing a class of counterexamples with squares and non-square rhombuses (Lexi and Wilson). For example, Wilson's written argument is shown below:

> By definition a rhombus is a polygon that has 4 equal sides that the angles add up to 360°. By definition a square has 4 90 degree angles with sides that are equal. A rhombus doesn't have to have 90° angles and a square does. Because of this 2 rhombuses don't necessarily have to be similar.

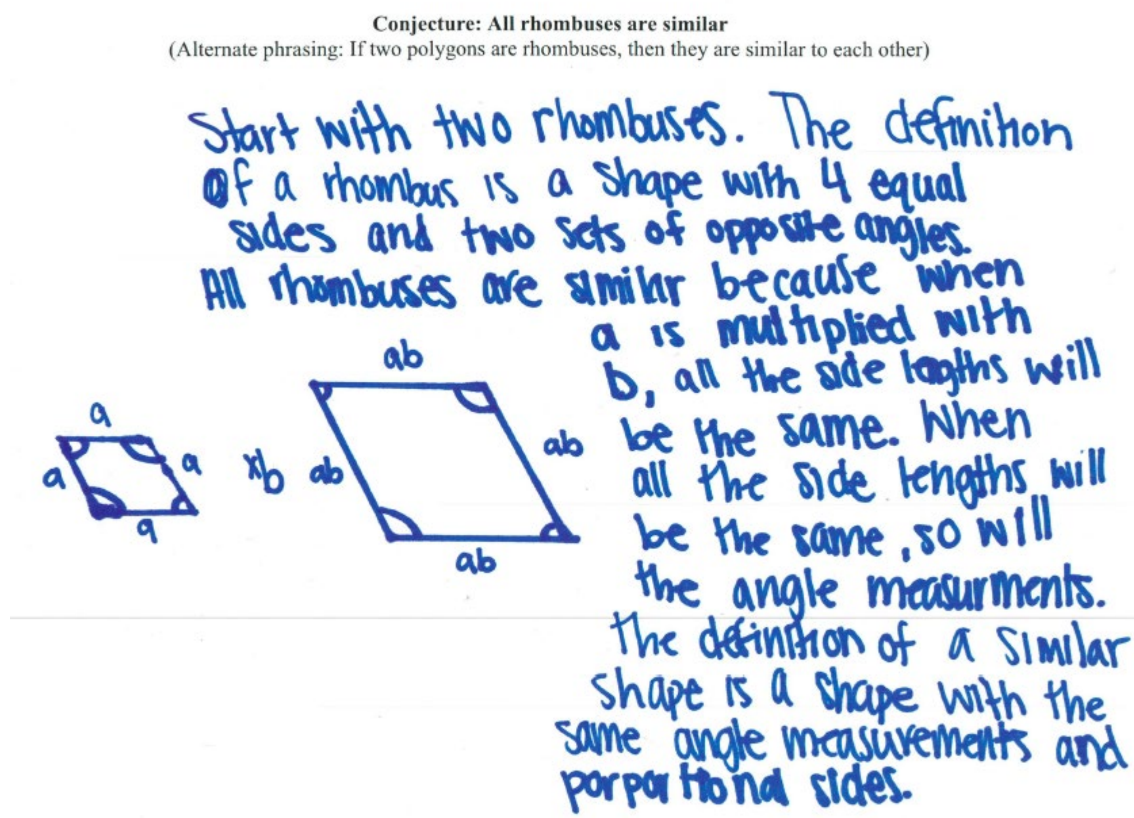
Wilson's argument demonstrates understanding that in order for the claim to be true, it must be true for all possible cases even though he did not disprove the claim with just a single counterexample. Whether by providing a single counterexample, or a class of counterexamples, to prove the claim was false, the six students demonstrated attention to the fact that the claim must be true for *all* rhombuses in order to be considered true.

The arguments produced by the remaining four students (Arin, Brian, Clay, and Heather), who initially thought the claim "all rhombuses are similar" was true, also demonstrated some attention to the domain of the claim. However, the way they conceived of the claim resulted in them considering only a subset of rhombuses. Arin, Brian, and Clay's arguments assumed that the angle measurements would stay the same as the sides proportionally changed, while Heather only mentioned proportional side lengths (not equal angle measures) when stating the definition of similar polygons. In order to illustrate how Arin, Brian, and Clay were thinking about the claim, we focus on Arin's argument, shown in Figure 5.

Arin appropriately defined a rhombus and stated the definition of similar polygons, but incorrectly claimed, "When all of the side lengths will be the same, so will the angle measurements." She verbally justified this claim saying, "if the shape's proportional, then it'll just… it'll like make the, um, the shapes more bigger, but the angle measurements will stay the same because the shape isn't changing its shape, it's just changing its size." This additional information suggests that she viewed one of the rhombuses as a dilation of the other. Instead of thinking about the conjecture as selecting two arbitrary rhombuses and then determining if they were similar, she appeared to be thinking about the task as selecting one arbitrary rhombus and then dilating it to create the second rhombus. When asked whether rhombuses have particular angle measurements, Arin stated that the opposite angles "have to be the same, but other than that, they don't have to be specific." This reply further confirms our interpretation that her belief in the claim's validity was based on her understanding of similarity and the domain of the conjecture. Of the four students who initially thought the claim was true, Arin, Brian, and Clay appeared to have at least a surface-level understanding of the definition of similar polygons (i.e., could state the definition), which suggests that their initial belief that the claim was true was not due to a lack of content knowledge. Instead, their initial assertion that rhombuses are all similar appeared to be rooted in how they were interpreting the domain of the mathematical claim, that is, how they brought to mind "all" rhombuses. Overall, students' work on the similar rhombuses proof tasks highlighted the abstract level of thinking needed to fully grasp what it means to prove that a general claim is always true and raises the question of how to support students in developing such understanding. We next discuss some of these points.

Figure 5

Arin's Written Argument for the Similar Rhombuses Proof Task



Discussion

By examining the enactment of general claim proof tasks with respect to students' opportunities to (1) engage in reasoning-and-proving activity and (2) consider the domain of the claims, this study extends Otten, Gilbertson et al.'s (2014) focus on the nature of mathematical statements found in reasoning-and-proving tasks in Geometry textbooks. With respect to the opportunities for reasoning-and-proving (RQ1), we found that the general proof tasks provided opportunities for students to actively engage in all of the intended reasoning-and-proving activities. Additionally, students also went beyond the intended activities by making conjectures/claims, posing counterexamples in response to a peer's claim, and refining a peer's conjecture during the tessellation tasks; drawing conclusions and making sense of the claims during the constructing diagrams task; and providing non-proof rationales and revising a conjecture during the proving similar polygons task. The tessellation task and proving similar polygons task in particular provided opportunities for students to engage in reasoning-and-proving in an integrated manner, mirroring the intent behind the hyphenated term (G. J. Stylianides, 2008). Across the three tasks, these students who were new to proof engaged in all of the reasoning-and-proving activity identified in Otten, Gilbertson et al.'s (2014) framework, including multiple opportunities to construct a proof. Consequently, these general proof tasks provided students with opportunities to develop their understanding of proof by engaging in the reasoning-and-proving process, something that Otten, Males, and Gilbertson (2014) noted was lacking within the introduction to proof chapters of many U.S. Geometry textbooks.

Although not a focus of the present study, opportunities for additional, unplanned reasoning-and-proving activity surfaced in part due to several factors, including the use of a launch-explore-summarize lesson structure (e.g., Lampert, 2001; Stylianou, 2010), which engendered opportunities for students to make sense of the tasks in small groups before being given more formal instruction. There was also a sense of at least partially shared authority, with the expectation that students consider and respond to their peers' ideas (e.g., the whole-class conversation in the tessellation task), rather than looking to the TR for validation. In the case of the tessellation tasks, the additional reasoning-and-proving occurred as students worked to make sense of two general claims ("do all quadrilaterals tessellate?" and "do all regular polygons tessellate?") that had parallel structure (i.e., both investigated whether a particular class of shapes would tessellate) but differed in validity. When considered together, the two claims motivated a need for a non-empirical justification that explained *why* only some polygons tessellate. The factors we have proposed that may have positively influenced students' opportunities for reasoning-and-proving align with the idea that opportunities to learn extend beyond the specific tasks given to students. Other factors have also been found to be important, such as "the emphasis teachers place on different learning goals and different topics, […], the kinds of questions they ask and the responses they accept, [and] the nature of the discussions they lead" (Hiebert & Grouws, 2007, p. 379). Given that prior classroom studies have documented instances where teachers began by modeling the proof construction process and retained most of the mathematical authority (e.g., Harel & Rabin, 2010; Martin & McCrone, 2003; Otten et al., 2017), future studies could analyze specific instructional features that facilitate opportunities for students to engage in reasoning-and-proving that extends beyond the opportunities within the original task.

Analysis of students' attention to the domain of the claims during the three tasks (RQ2) highlighted the complexity of addressing generality and the different ways it impacts the reasoning-and-proving process. Specifically, attempting to consider all possible cases when investigating the validity of the claim (tessellation tasks) required a different shift in attention than depicting the generality of a claim when constructing and notating a diagram (constructing diagrams task) or proving a claim to be true for all possible cases (proving similar polygons task). In all three cases, it seemed to be important that the claims themselves were general, as opposed to an introduction-to-proof unit that presents simple, particular proofs (e.g., "write down the justifications for how we know that this segment of the given diagram is congruent to this other segment"). Although the general proof tasks afforded certain opportunities, as discussed above, they worked in concert with other factors such as the TR questions and the interactive dynamics. Moreover, the transfer of any attention to generality is not guaranteed, evidenced by the varied attention to generality during the similar rhombus task (final interview) despite everyone attending to generality during the similar squares task (Session 11).

We also wish to comment on the attention to the domain of claims over time. Towards the end of the study, students' work began reflecting an increased attention to the domain of the claims. We viewed this as a positive development as this was students' first formal introduction to proving. Yet the attention to the domain near the end of the study was nuanced. Whereas all student work demonstrated at least some attention to the domain of the claims in the proving similar polygons task, some students interpreted the claims in a way that only encompassed a subset of cases (e.g., Arin's argument in Figure 5). In the case of Arin, subsequent conversation suggested that her initial (incorrect) belief that all rhombuses were similar was not a result of a lack of content knowledge, but rather how she was thinking about the claim itself. Although Arin's work does not discount the role that content knowledge and proof skills play in understanding why some students construct empirical arguments for general claims, it does reinforce the particular difficulties students face in interpreting diagrams as figural concepts (Fischbein, 1993) and the need to better understand how students interpret the domain of mathematical claims (Mason, 2019).

Given this study's small sample size involving accelerated students and the explicit emphasis placed on the domain of the claim by the TR, more research is needed to ascertain the extent to which a wider range of students recognize the domain of the claim while engaging in reasoning-and-proving tasks. It is possible that students' prior successes in mathematics and their involvement in the accelerated mathematics program may be a form of selection bias contributing to the findings of additional, unplanned forms of reasoning-and-proving. That said, the accelerated program had only focused on algebraic topics at the time of the study, so the students' content knowledge was likely not significantly different from other students at the school as they began studying secondary geometry. Future research should involve students who were taught using a more traditional curriculum where particular claims are frequently used (Otten, Gilbertson et al., 2014) and the domain of the claim is often obscured through the use of separate "given" and "to prove" statements (Chazan, 1993).

How should we view students' work on the three tasks, given that they were completed as they were first being introduced to proof? Viewing students' work in the teaching experiment solely through the lens of the enacted opportunities to learn (RQ 1) paints a rosy but incomplete picture of their developed understanding of proof and ability to construct deductive arguments. On the one hand, there was evidence of careful attention to the generality of claims in nearly all of students' constructed arguments during the proving similar polygon conjectures task and on proof tasks given in the final interview (Conner, K.A., 2018). On the other hand, findings on students' attention to the domain of the claims (RQ 2) portrayed a more nuanced picture, in which students demonstrated attention to the domain of the claims in some instances, but not in others. In both the constructing diagrams task and proving similar polygons conjectures task, at least one group used specific numbers when labeling aspects of their diagram that can vary, even in instances when they explicitly referenced the domain of the claim in their language. One way to interpret students' inconsistent shifts in attention is to conclude that they developed understanding of generality at a surface level (i.e., they recognized the types of justifications that were appropriate and the need to prove the claim for all possible cases), but had not fully become aware of other aspects of proof that are impacted by it. Nevertheless, for an introductory unit, the fact that general proof tasks helped set the stage for a focus on the domain of claims and students began to discuss those domains, even if imperfectly, it may be a sufficient foundation on which to build. As the field continues to explore ways to improve the teaching and learning of proof, more research is needed on ways to support shifts in attention with regard to the level of generality indicated within a mathematical claim, and its impact in the reasoning-and-proving process.

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Appendix A

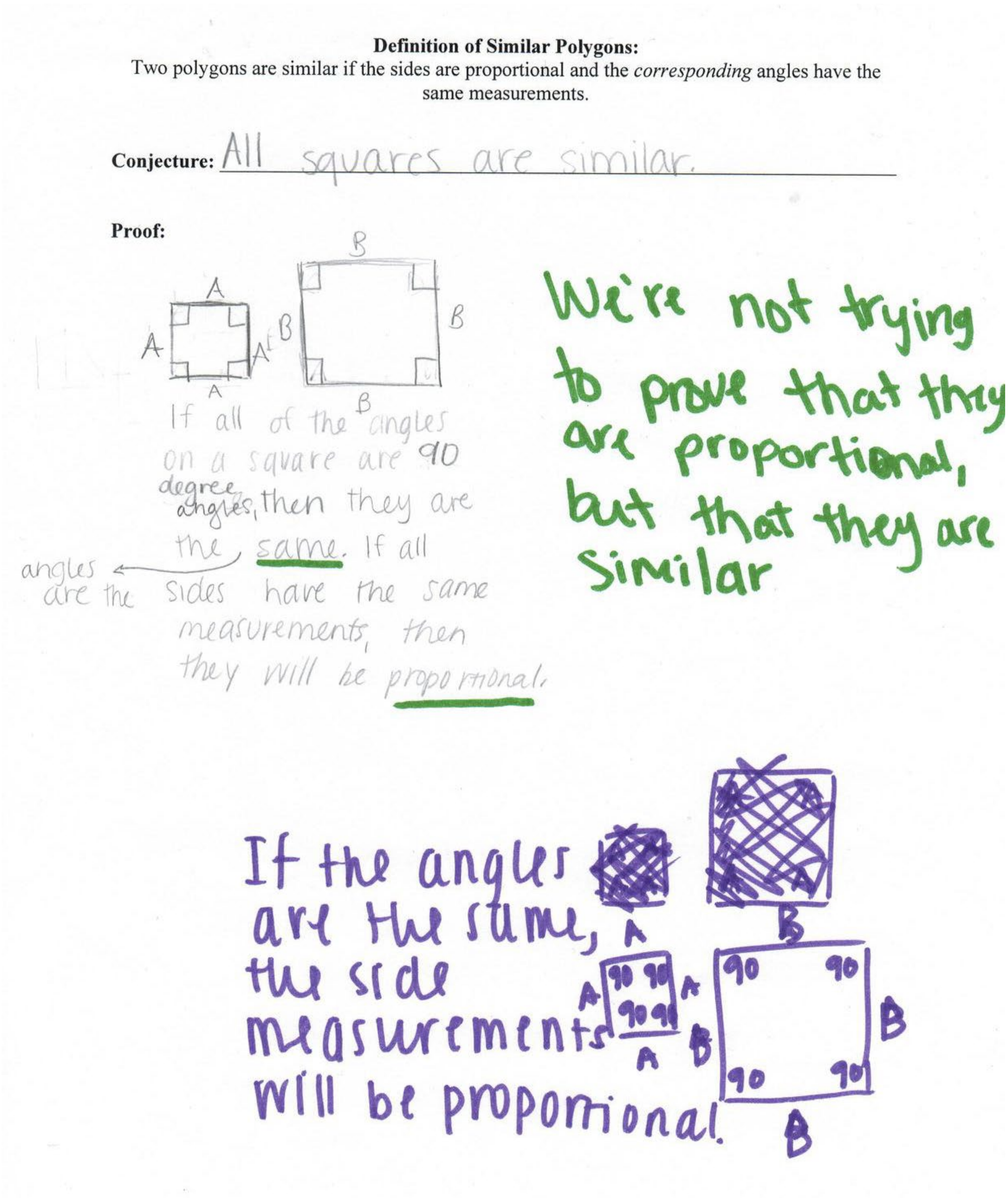
Overview of the Instructional Sequence

Session	Classroom Activities	Rationale for Tasks
1	"Do all quadrilaterals tessellate?"	Allowed for testing of specific cases where the cases seem unique due to differences in the diagrams. The task proof allowed for explanations of *why* it was always true.
2	Create step-by-step directions that explain how to tessellate *any* quadrilateral.	Aimed to facilitate systematic work and the identification of cross-cutting features of quadrilaterals that result in the figure tessellating.
3	"Do all regular polygons tessellate?" "Do you still think that <u>all</u> quadrilaterals tessellate? If no, explain why. If yes, is there something special about quadrilaterals that make it so that they will always tessellate?"	First question served as a pivotal counterexample (Stylianides & Stylianides, 2009) to cast doubt on their prior confidence that all quadrilaterals tessellate based on checking specific cases. The second question emphasized that a claim must be true for *all* cases and motivated the determination *why* a polygon will or will not tessellate.
4	Summary of first three sessions; TR explained why all quadrilaterals, but not all regular polygons, tessellate.	Introduced the explanatory feature of proofs.
5	Circle and Spots problem and Monstrous Counterexample (Stylianides & Stylianides, 2009)	Aimed to cast doubt on the idea of using examples to determine whether a statement is *always* true.
6	Introduce generic examples through exploration of a number trick: https://nrich.maths.org/2280. Next, students explored and proved: "9*11 equals 1 less than10^2, 3*5 equals 1 less than 4^2. Will this pattern always be the case?"	Aimed to support students in interpreting and using geometric diagrams where they only attended to the features that extended across all cases within the domain. The second task aimed to facilitate interpretation and use of variables as varying quantities.
7	Students constructed diagrams for six quadrilateral theorems	Introduced conditional statements, notation methods, and what can/cannot be assumed true based on a geometric diagram.
8		
9	Develop definition of similar polygons; based on sequence in Kobiela and Lehrer (2015)	Established necessary mathematical content knowledge for Sessions 11 and 12.
10		
11	Students posed conjectures of the form "all ____ are similar", drafted an argument for squares, critiqued peer arguments, revised their	Developed understanding of proof by engaging in multiple aspects of the reasoning-and-proving process in small groups. The

12	arguments, then discussed proof as a whole class. Next, students investigated remaining claims from previous sessions.	second session was used to introduce specific characteristics of proofs.
13	Students individually engaged in the reasoning-and-proving process (described in sessions 11-12) for the exterior angle theorem. Task was posed using two examples, followed by the question "is this a coincidence?"	Developed understanding of proof by engaging in the reasoning-and-proving process. Individual written work was used as a formative assessment.
14	Students developed shared criteria for features of "good proofs"; task based on Boyle and colleagues (2015).	Assessed conceptions of proofs and reflected on key ideas from the teaching experiment.

Appendix B

Brian, Arin, and Sadie's (Group 1) revised argument for the conjecture, "all squares are similar", including the feedback given by Group 2 (bottom) and Group 3 (top right).



Definition of Similar Polygons:
Two polygons are similar if the sides are proportional and the *corresponding* angles have the same measurements.

Conjecture: All squares are similar.

Proof:

A A A A B B B B

If all of the angles on a square are 90 degree angles, then they are the same. If all sides have the same measurements, then they will be proportional.

angles are the

We're not trying to prove that they are proportional, but that they are similar

If the angles are the same, the side measurements will be proportional.

A A A A 90 90 90 90 B B B B 90 90 90 90

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Conversations with Scientists and Science Educators: In Search of the Third Dimension

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ABSTRACT

We have argued that science in general, and scientific inquiry in particular, is a human activity and that current models to describe science (either as Scientific Method or as a Body of Knowledge) tend to underestimate the significance of human beings in the phenomenon. Our earlier paper (Bevins & Price, 2016) suggests a model to correct this imbalance which we call 3-Dimensional science. The current paper used the Storyline Method to look at the lived experiences of nine science researchers and educators for evidence of our third dimension (Psychological Energy). Results suggest that the key features of our third dimension are present, namely: a degree of autonomy, a sense of competence, and a relatedness to significant others. We suggest this further strengthens the argument for a more holistic approach to science education which celebrates these issues rather than simply a technical analysis of isolated teaching techniques.

Keywords: inquiry, nature of science, scientific method, science education

Introduction

Science is broadly-defined in terms of theories, facts and practises. The theories and facts can be described as a single dimension of science which we call Dimension 1 (D1). D1 contains facts (e.g. the melting point of sodium, the atomic weight of hydrogen) and potentially complex, ideally general, theories (e.g. natural selection, kinetic theory) that organise these into productive epistemic structures. Even though the exact contents of D1 are open to discussion there is general agreement, for example, that the melting point of sodium is 'in' while the names of the Kings and Queens of England is 'out'. Over the last 50 years, the growth in D1 has been significant with whole disciplines being created (e.g. the nature and management of genes). This growth in D1 is delivered and regulated by a set of practises that we describe as Dimension 2 (D2). D2 includes rules concerning the collection, and analysis of, evidence by scientific method typically involving generation of predictions and hypotheses which are tested through experiments. Increasingly D2 includes skills like networking and communication required to operate within the modern, global scientific community. In summary, D2 includes enabling skills (e.g. networking, communication), inquiry skills (e.g. control of variables) and mechanical skills (proficiency in specific laboratory procedures). The dimensions, although related, vary independently. It is possible to have a strong D1 (you might know lots of facts and theories) but be deficient in D2 (weak communication skills). Similarly, a strong grasp of the key skills of D2 may not always indicate strong grounding in D1.

Much of the discussion in science education has concerned itself, not always helpfully, with the emphasis placed on these two dimensions with 'knowledge-rich' tradition typically favouring D1 and the adherents of a more process-led approach emphasising D2 (Hmelo-Silver et al., 2007; Kirshner et al., 2006). However, even if the perfect balance could be agreed, the two-dimensional model only covers science as a disembodied, crystallised entity. It produces a portrait of existing science knowledge alongside a statement of the rules of engagement rather than reflecting science as it is practised across the world. We have argued (Bevins & Price, 2016) that a better model of science requires a third dimension. We call this improved model three-dimensional science or 3D science.

Three-dimensional Science

We have described our model for three-dimensional science in detail elsewhere (Bevins & Price, 2016) and so provide only a summary here. 3D science includes three dimensions;

- **D1 A body of knowledge**: this informs scientists' thinking about phenomena and can generate questions and suggestions for inquiry.
- **D2 Evidence-management procedures**: these ensure evidence is generated reliably, interpreted with reference to the underlying ideas and the observed data, and communicated appropriately.
- **D3 Psychological energy**: this provides the energy to create and manage a scientific inquiry.

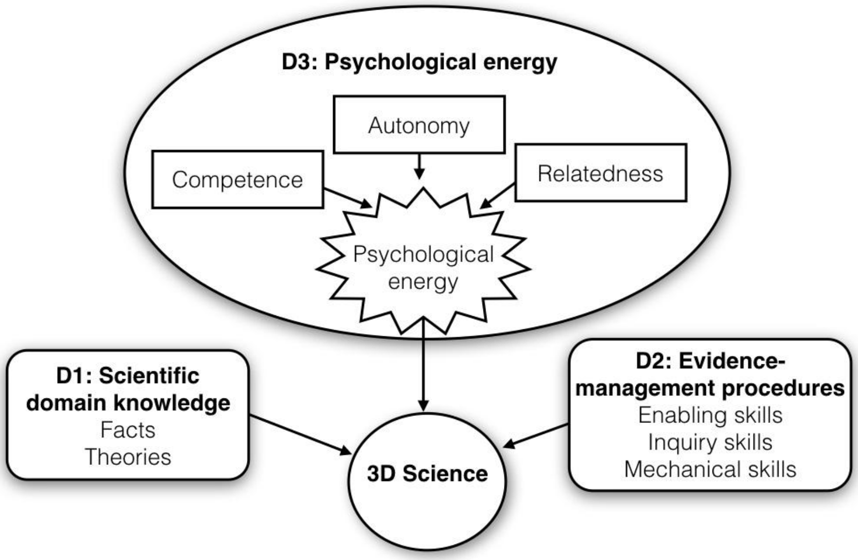
These dimensions have different natures and characteristics and do not link conveniently to each other in a simple sequence. One does not 'lead' to the other nor 'depend' on another in a strict linear sense. All are interrelated but only to the extent that they belong to a system that requires their presence. We have called this model a 'fruit salad' model in that the dimensions are as related to each other as the individual fruits in a fruit salad. They are all essential to the makeup of the salad, but apples are not like bananas and pineapples do not lead to oranges or grapes. The system is more than merely the sum of its parts even though the parts might be externally still recognisable. Figure 1 provides a visual summary of the components of our 3D model.

Understanding D3

Dimension 3 (D3) of our model provides the energy for scientists to operate the machinery of D1 and D2 to drive the further development of the scientific domain. A useful analogy might be to think of D1 as the written-down steps of a ballet, D2 as the ability to complete them through repeated, often formal, exercises while D3 is a feature of the dancers themselves that convert these written steps and practised movements into an actual performance that has meaning and integrity. Without the dancers there is no dance. This sees D3 as a feature of the active scientist, their motivation, commitment and sense of purpose, which drives their engagement. D3 varies from low engagement with limited personal purpose (low energy), through to a clear personal purpose and engagement (high energy). We feel that the 3D model offers a number of advantages when thinking of scientific activity and we describe these in the discussion section that follows.

Figure 1

The 3D Model of Science



Self Determination Theory's (SDT) and the Third Dimension

We draw on Self Determination Theory's (SDT) treatment of motivation (Deci & Ryan, 2012) to inform our understanding of the nature of D3. As Deci and Ryan (2006) explain: "it (the motivation to act) must be endorsed by the self, fully identified with and "owned"" (p. 1561). The endorsement requires a degree of autonomy in the actor since something that is forced upon them cannot be 'owned' - it is, by definition, an imposition 'owned' by an external. Similarly, a person should also feel a degree of competence in the task; in effect the task is appropriate for them and their skills - there is a 'good fit' between act and actor. Finally, the task must in some way be valued by others who are significant for the actor. This is the notion of relatedness central to SDT's understanding of motivation.

Why Another Dimension?

We understand that our use of the 'third dimension' can cause confusion as we are not the first to use the term. The Next Generation Science System (NGSS Lead States, 2013) in the United States talks explicitly of 'three dimensional science', although they mean a particular subset of generally applicable concepts like 'patterns', 'cause and effect', 'structure and function' or 'stability and change' by the third dimension. These are valuable ideas that share some similarities with the concepts of evidence (Gott et al., 2008) and identifying a separate 'third dimension' for these concepts is a useful way to draw attention to them. However, our model assumes they fit more naturally in Dimension 1 where they can impact and inform some of the activities in Dimension 2.

It is important to show how our third dimension links into existing discussion around the 'Nature of Science' (NoS). The understanding of the NoS is not fixed, showing a shift from the notion of science as an external, logical process of justification, producing objective, value-free knowledge to the recognition that all observations are, to some extent, theory-laden (Khalick & Lederman, 2000). An understanding of the NoS is an essential part of science education, particularly with relation to the development of science literacy (Archer & DeWitt, 2016) for citizens and the ability for them to engage with important socio-economic issues raised by technological advances (Tala & Vesterinen, 2015). A review of the elements of NoS in teacher education courses is shown in Table 1 (Noushin et al., 2021).

Table 1

Elements of the Nature of Science (NOS) Covered in the Teacher Education Programs

	Example of NoS Elements covered in the teacher education programs
1	Scientific knowledge is based on *empirical* evidence.
2	*Society and culture* influence each other with respect to science.
3	Scientific investigations are influenced by theory and by scientists' backgrounds, and therefore, *subjectivity* is part of science.
4	*Creativity* plays an important role throughout scientific investigations.
5	Both *observation* and *inference* are important in the construction of scientific knowledge.
6	Scientific knowledge is durable yet *tentative*.
7	Science uses *shared methods* and there is not a single scientific method.
8	*Scientific theories and laws* serve very different and not interchangeable functions.
9	A scientist works within the *scientific community* to evaluate and contemplate the work of other scientists.
10	The evidential part of science is strongly advanced by the *technology* available at the time.
11	Science and *religion* are different ways of knowing.

While the NoS elements may not concern themselves with the melting point of sodium or the theory of evolution, they are claims based on evidence in the same way as Newton's claim that for every action there is an equal and opposite reaction. They are a valuable part of science and science education but they do not require a new dimension - they can be accommodated with Dimension 1 in our model.

Reviewing issues concerned with the teaching of the NoS, Bell (2009) identifies three domains which contribute to the creation of science, these are a body of knowledge (equivalent to our D1), a set of procedures (equivalent to our D2) and what he calls 'a way of knowing'. The 'way of knowing' includes a set of statements (e.g. 'Scientific knowledge is based on evidence', 'Creativity plays an important part in science' which help to describe science as it is practised). These statements give rise to a set of what he calls 'key concepts' (e.g. the tentative nature of scientific knowledge). Taken alongside the other domains these concepts describe the NoS. Again, these can be sufficiently described in terms of concepts and procedures, although these concepts and procedures may not, at first glance, appear in traditional content lists for science courses or be unique to science.

A Testable Claim for D3

In comparison to NGSS' third dimension or even the material on the NoS, our D 3 is not a selection of valuable concepts, insights or attitudes or even habits of mind (Gauld, 2005). It draws on many of the concepts mentioned in the discussion of the NoS and will be manifested in some of the activities that fit easily into D2. Our D3 is concerned with the psychological energy to drive the processes of D2 which generate, and apply, the concepts of D1. In physics education, energy is a notoriously difficult concept to teach, perhaps because the many 'forms' of energy (e.g. sound, light, heat) are not energy itself but are the *effects* energy creates as it transfers within and between systems. The energy itself is invisible. In the same way, our psychological energy is invisible but manifests in activity, and in the context of science this means scientific research. Again, as in physics lessons, a key issue for energy is where it comes from and what happens if the supply runs out? The energy that resides in D3 comes from the individual scientist and we explain below how Self Determination Theory has helped to inform our thinking on this. A collapse in energy supply (D3) means scientific research stops; the body of knowledge (D1) remains intact but static, the skills and procedures (D2) pristine, but unused.

So, if D3 is both real and necessary for a complete description of science, we should be able to find evidence of it in the way scientists describe their scientific activities. Our model predicts that when there is clear evidence of all three dimensions present the scientist should be engaging in work that is demanding but rewarding, objectively significant and personally satisfying. A gap in any of our three dimensions should inhibit this productivity. This provides us with a clear testable claim which we explore in our data collection: if people are actively engaging in scientific activity there should be some evidence of their D3 needs being met.
This claim generates two related research questions:

- Can we find any evidence of D3 needs being met in scientists' accounts of their educational careers and professional lives?
- Is there any suggestion that D3 is not merely helpful but essential, i.e. when D1 and D2 needs are met but D3 is absent does the science slow down or stop?

To explore the experiences of scientists over their lives and to look for evidence of the three dimensions we developed an approach using Storyline method (Beijaard et al., 1999)

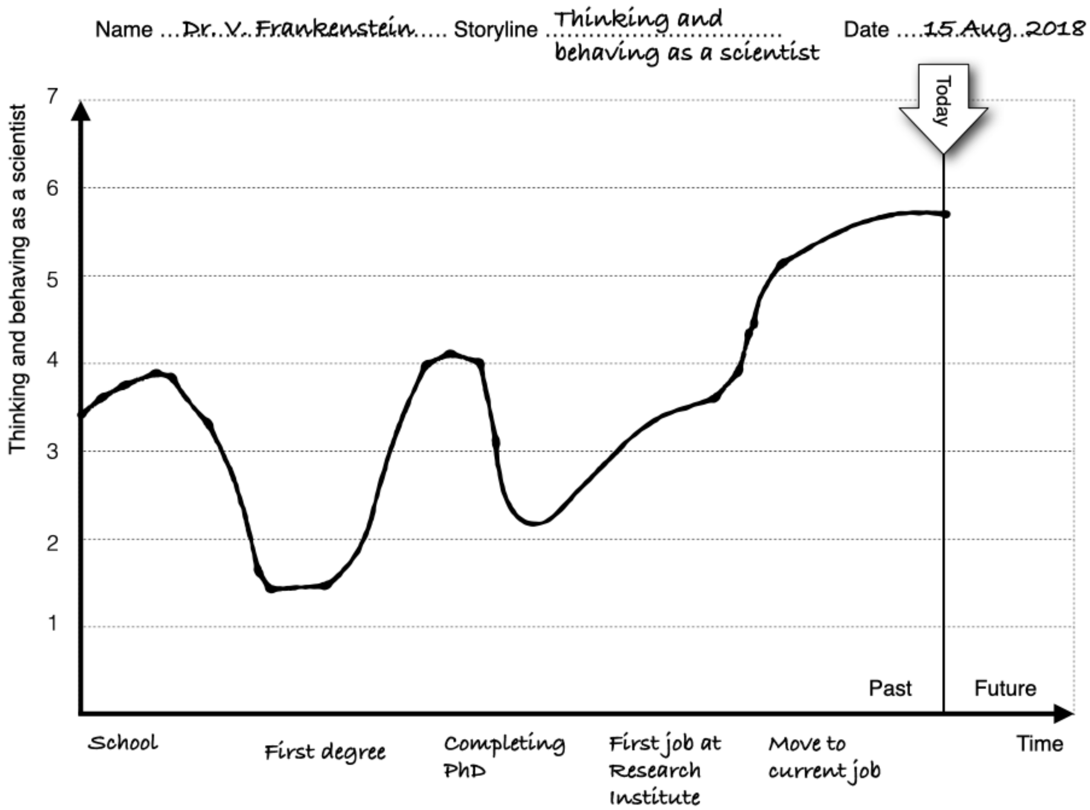
Methods

We used the storyline method (Beijaard et al., 1999) to stimulate and frame conversations with practising scientists. The participants were given a chart containing a pair of axes with the present day fixed at the far right of the x-axis. Each participant was then asked to score their current level of 'thinking and behaviour as a scientist' and mark it on the line labelled 'Today'. They were then asked to draw a line backwards through time showing rises or falls in their scientific activity. Labels could be added to the x-axis to identify significant events or periods. Figure 2 shows a typical storyline plot.

Once participants completed their storylines, we engaged them in a conversation to explore reasons for these rises or falls. This focus on their own story places the participants in a relatively powerful position and, from experience, they are both motivated and skilled in their analysis as they explore their understanding of what behaving as a scientist means to them.

Figure 2

An Exemplar Completed Storyline Form



Data Analysis

Each conversation was audio recorded and transcribed. We used thematic analysis (Braun & Clarke, 2006) to identify patterns and themes within our transcriptions. Themes represent something important about the data related to our original research focus (e.g. the factors impinging on their lives as scientists) and provide some level of meaning (e.g. obtaining research funding, designing an experiment). The three dimensions became our superordinate categories in which we housed themes. The transcribed conversations were read and re-read as we developed notes prior to agreeing on themes to be placed in each superordinate category. Throughout this process we engaged in reflective discussions to ensure the specifics of each theme appropriately represented a feature from one of the dimensions. This 'theoretical' approach framed our analysis in contrast to a purely inductive approach more typical of Grounded Theory (Charmaz & Belgrave, 2015). Thus, by asking the participants to explain what they meant when they said they were 'acting as scientists' we could analyse their perceptions of the nature of science which in turn, allowed us to identify references to the three dimensions that we claim our 3D model of science presents (see Table 2).

Sample

We identified eight scientists (three female, five male) from India and the UK, working at university level, who had experience as practising scientists and educators. The participants formed a convenience sample (Etikan et al., 2016) and were identified through ongoing project links.

Table 2

Indicators of the Presence of the Dimensions

D1: Scientific domain knowledge	D2: Evidence-management procedures	D3: Psychological energy
References to scientific domain knowledge (D1's facts and theories). Conversations should also emphasise increasing levels of scientific understanding (e.g. new subjects, higher levels of treatment).	References to practical work (mechanical skills are part of D2) but also the notion of designing experiments, the 'control of variables strategy' (Schwichow et al., 2016) which appears in D2 as inquiry skills and working in teams (D2's enabling skills).	References to having a degree of control over the work both in terms of its purpose and implementation (autonomy). This offers the option to engage and develop it because it is in harmony with the scientists' views, values, and perceived abilities (competence). There will also be references to significant others who have contributed to the scientists' choices (relatedness).

Ethics

Each participant was briefed about the research purpose, potential outcome and dissemination. They were informed that all data would be anonymised and stored on an encrypted drive within the university. All participants were made aware of their right to withdraw from the research at any point and that their data would be destroyed immediately. Consent forms were gathered from all.

Findings

This section describes the interviewee's understanding of the nature of scientific endeavour based on data gathered through conversations and storyline graphs. Background information not directly related to science (e.g. family history, strong peer group friendships) from the interviews is included where relevant. Reproductions of the storylines drawn are included with markers [numbers in square brackets] to indicate where during the interview a particular point was emphasised. We also offer specific quotes with the relevant timecode. The data are presented by individual conversation and general observations covered in the discussion section which follows.

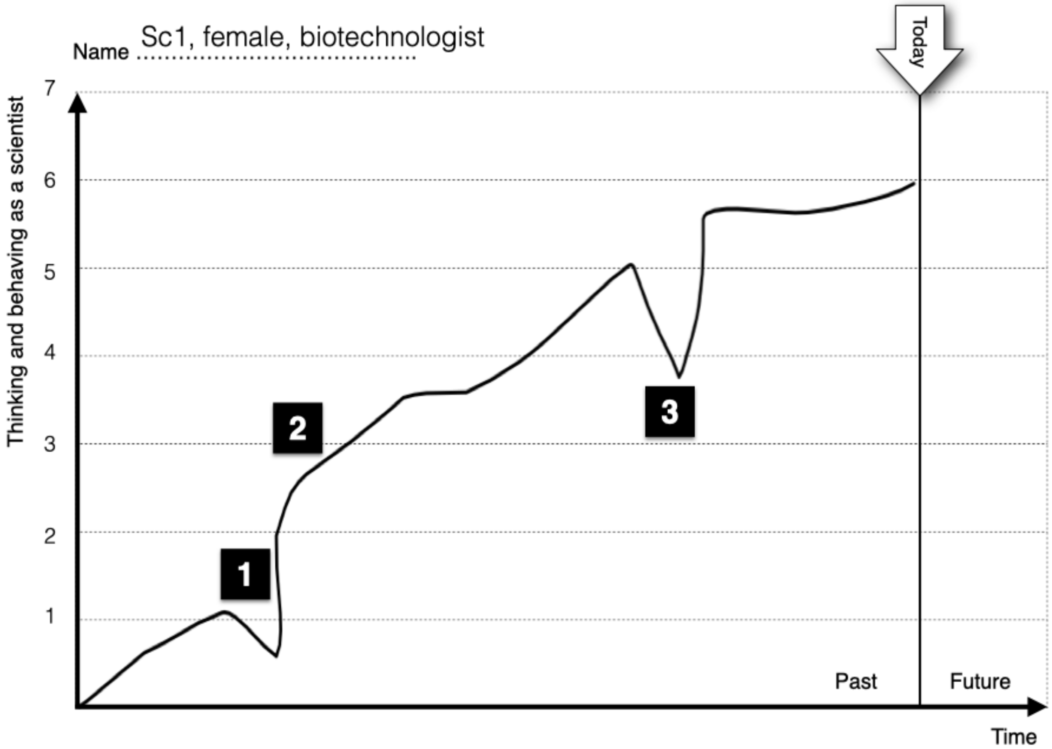
Sc1, Female, Biotechnologist

Much of Sc1's conversation concerned her experience of learning and little was positive, the first dip on her Storyline [1] was as she entered college (age 18), where science was being taught in a very didactic manner:

basic biology was very good but mode of learning was not … (Sc1/ 9:52)

Figure 3

Sc1's Completed Storyline Form



Afterwards she joined a new course that involved much more practical work and decision-making by the students [2]. The references to 'practicals' and 'we were doing it ourselves' correspond to D2 while her personal involvement in the process (D3) is clearly referenced in her explanation that when they made mistakes they felt bad about it:

> In our Masters we were given a lot of practicals to do… a lot of exposure to hands-on training. The teacher would not demonstrate the practical…we were doing it we were making mistakes and feeling bad about it. (Sc1/12:50)

The centrality of practical competence (D2) was emphasised in talk of 'good hands' (practical skills) which she enjoyed (D3):

> In scientific research people say that she or he has very good hands … technical hands … so a lot of technical skills were imparted to us … which we enjoyed. (Sc1/13:02)

However, the lab activity began to pale as it became more repetitive and focused around a small, technical problem. She explained that she was not happy finding an unknown protein and working on that for the next ten years. She talked of work becoming a mundane trudge for more data. This disenchantment with the routine of lab work produced a significant fall in her storyline trace [3]. In these comments we see the effects of a relatively low D3 - Sc1 had felt purposefully involved and doing something of substance early on in her career (high D3) but as the work became more mundane and repetitive (low D3) it lead to her leaving the profession and moving into a full time teaching role.

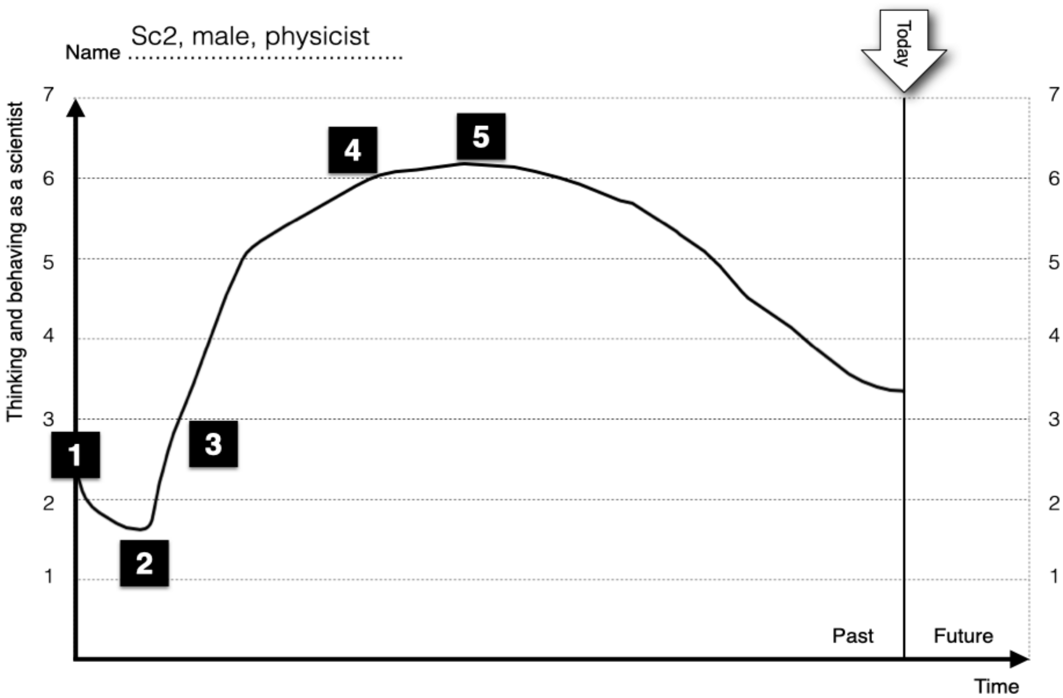
Sc2, Male, Physicist

Sc2's obvious warmth when recalling his friendship with other students at his school [1], who were also interested in science, resonates with D3's emphasis on 'relatedness'. The subsequent lack of similar enthusiasts at first degree led to a fall [2] in his feelings of being a scientist:

> I was very unhappy with that (no friends who were similarly enthusiastic) … during my BSc … even the teaching methodology was more memory-orientated than problem-solving … that was very disappointing (Sc2/9:23)

Figure 4

Sc2's Completed Storyline Form



The situation improved dramatically during his MSc [3] where the emphasis was on him personally taking the initiative to solve problems - another aspect of D3:

> There was no memory-oriented tests it was all problems … so, the teacher will teach you concepts and you'll work on it, try to understand it more on your own and the exams would be completely new problems. (Sc2/10:00)

He described his BSc as 'training to do science' not 'doing science', whereas in his MSc work and post as a Project Assistant [4] he was working with new ideas, (D1) exotic materials and lasers (D2). There was a strong sense that high levels of domain knowledge at the edge of evolving theory (D1) and the use of complex equipment and sophisticated techniques (D2) were essential for his sense of 'being a scientist' [4]. The peak of this storyline was during his PhD [5] when he had much more personal control (D3) over his work. Sc2 reflects the experience of other participants who felt that they can only do what many call 'real science', in distinction to merely following instructions, when they have control (D3) of their work during their PhD years or when working on a project with a team that can make its own decisions. Typically they will also be deploying sophisticated skills (D2) and using complex ideas (D1) at this time.

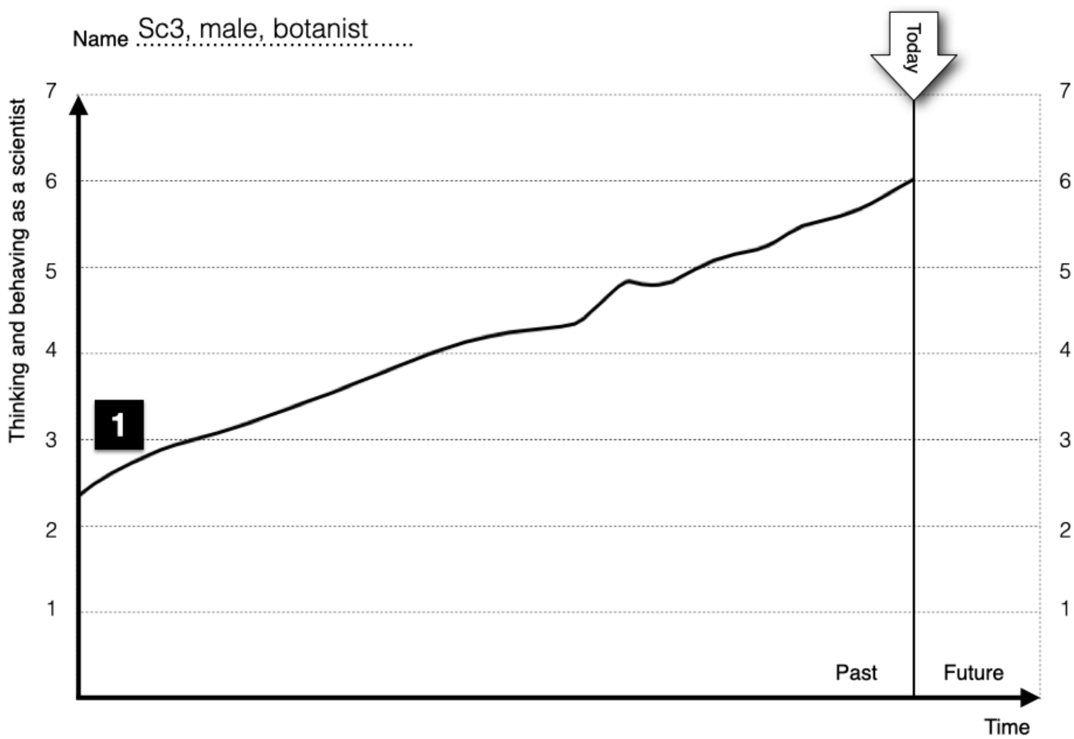
Sc3, Male, Botanist

Sc3 discussed his choice to become a research scientist in terms of finding solutions to 'help society' [1].

> That was a time when I had to decide if I was going to continue with my higher studies or if I was to go for a job. … I decided I could continue and look for certain problems and solutions which may help society itself. (Sc3/2:15)

Figure 5

Sc3's Completed Storyline Form



He explained scientists as people who find solutions to societal problems - because they are able to think systematically and scientifically:

> I think it was thinking as a scientist … you have to look at the problems in a scientific manner…they can be solved in a better way…in a scientific manner. (Sc3/3:07)

These two anecdotes contain evidence for D2, 'looking at the problems in a scientific manner' but also show a strong involvement of D3 in talk of doing something to help 'society itself'. The importance of working in a way that makes a contribution to a range of people (SDT concept of relatedness) is strong evidence of the existence of D3, referenced again later in the conversation:

> I thought that was necessary for the survival of everyone on this planet because if you do not work in a scientific manner the system may collapse. (Sc3/5:15)

Returning to the issue of the gradual rise in his storyline trajectory we asked what was different in his 'scientific thinking' at 16 years of age and now that he was 20 years older. His response was of a gradual increase in knowledge and skills - developing D1 and D2:

> I think its the constant learning…because when I was at school I was not exposed to many things there…I think its the exposure… how you have been exposed to problem-solving capabilities. (Sc3/7:18)

He saw his progress in terms of being more autonomous and more 'self guided' (Sc3/11:49) and when he added this to his storyline he claimed this extra capability developed during his MPhil and subsequent PhD studies.

Sc4, Male, Biologist

Sc4's storyline covered his whole life which gave him a chance to describe the difference between knowing 'how science works' (D2) and a sense of science as an approach to life in general, which he referred to as 'scientific temperament':

> I am a teacher, I am post-doc who knows how science works. I do know the components that some sort of observation is there, then we do have some sort of hypothesis…we do test these things, and then we draw some conclusion and theory. (Sc4/6:50)

However, his personal curiosity, and drive to ask questions, which seems more like D3, was also active. He makes this clear when he compared his PhD [2] with his preschool life [1], where he knew none of the mechanics of scientific method, but felt he was much better at thinking like a scientist:

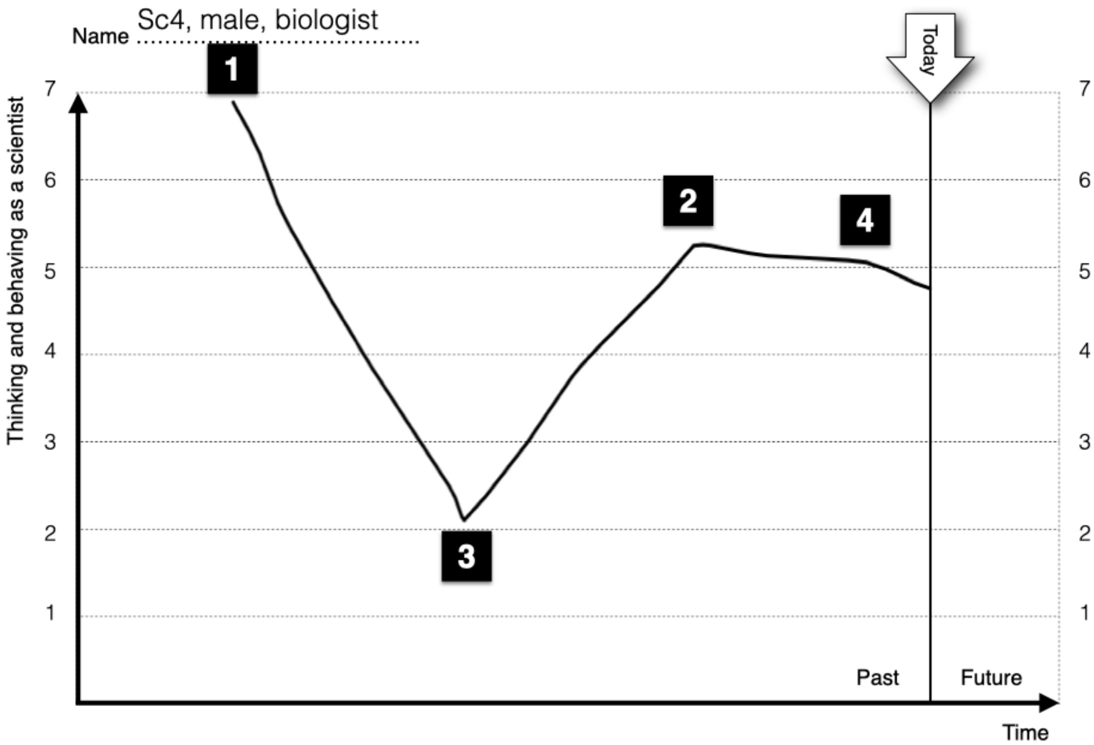
> If I compare this phase [2-4] to preschooling or the first five years of my life it was not taught to me these are the components of science as such…my observation power was more. I was questioning each and everything, no matter what resources were available or not but I was testing it right? (Sc4/6:57)

He made this distinction between knowing technique and intrinsic curiosity very clear with the statement: "It was not a formulated science but I was doing it." (Sc4/7:23) Since Sc4 recognised the attitudes of science in himself even prior to school it might have been hoped that when he went to school the formal rigour of scientific method would increase his ability and opportunity to think like a scientist. In fact, as he explained, this went down dramatically:

> When I came to the schooling phase [3] it dropped down tremendously because I was not given or provided that autonomy to think or question things….It dropped off because what has to be taught was fixed and how it has to be taught was also fixed so there was no autonomy for me so I stopped questioning. (Sc4/8:01)

Figure 6

Sc4's Completed Storyline Form



Significantly when Sc4 felt his autonomy (D3) was denied he did not feel he was behaving like a scientist. Indeed, much of Sc4's conversation suggested that science was an attitude of mind, he called it a 'scientific temperament' which revolved around making observations, asking questions, and trying things out without fear of censure in a methodical and organised manner (D2). His complaint about school science was that it offered no chance to question. Here he draws a distinction between a technician who follows procedures designed by others and a scientist who has the right (D3) and capability to ask novel questions and explore ideas and fields that are personally important to them:

> If you don't have the capability of observing, questioning, and analysing I don't think that person can be a scientist. There's a difference between a technician and a scientist I think. (Sc4/13:24)

During the last few minutes of the conversation he remarked about the slight fall in his scientific research as he took on more responsibility for teaching [4] - a common remark amongst other participants indicating a degree of loss of control of his personal timetable (D3).

Sc5, Male, Botanist

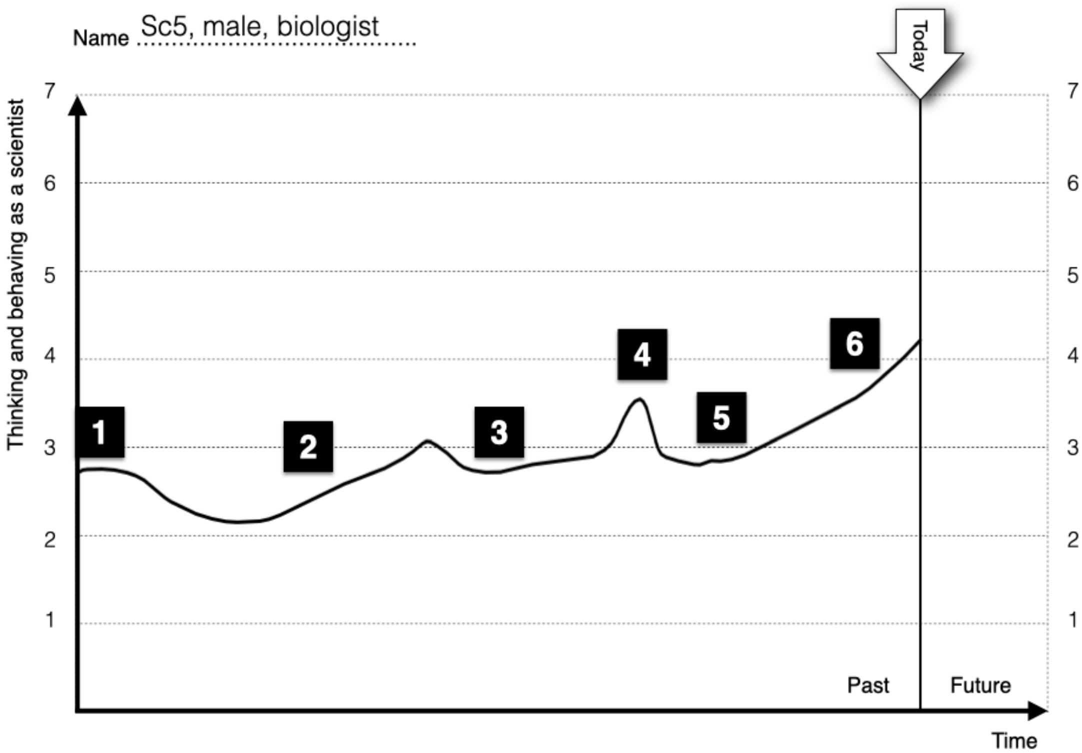
Sc5 started his conversation with his personal history:

> My parents are involved in agriculture…he (his father) would always tell me 'mangoes come this season' and I would ask him 'why? Are there mangoes which come in all seasons?' So this type of behaviour was there when I was young [1]. So he would get different types of mango plants and say 'Let's see which one comes first' and then the first ones would not be

tasty … that would make me wonder why these ones were not tasty but after rain they get tastier. (Sc5/5:00)

Figure 7

Sc5's Completed Storyline Form



Sc5 equated inquiry with curiosity and asking questions - with a real purpose behind the questions - even if only for a good supply of tasty mangoes! Notably his father helped him as he tried to grow different types of mangoes. His school also seemed supportive:

> …this kind of thinking (questioning and experimentation) was there. So it would connect to us when school projects [2] were made and then you could very easily take five mango plants and explain in school why this is sour and this plant is sweet. (Sc5/5:30)

Sc5's storyline shows only a small increase over time - largely based around original research. In his comments it is possible to see D2 (trying different mangoes and testing a hypothesis about mango and rains) and strong D3 (his supportive father providing a powerful 'relatedness', while his school offered options for projects) at his earliest age. The preparation for university [3] was more fraught with greater emphasis on D1 capability (the main feature of the assessment systems that controlled his entrance to university) coinciding with less laboratory work (D2) and, to some extent, others setting his career goals (low D3):

> So, the push (to agriculture) was good by the parents but in that time they (the schools and university-preparation institutions) were giving me no hands on in the lab. (Sc6/7:14)

In university [4] he spent more time in the lab (D2) and working on projects that were important to him. He got in well with his supervisor who was active in research. This illustrates an aspect of D3 - the need to work with significant others, the SDT idea of 'relatedness'. He explained any small dips in his storyline [5] by 'other pressures', he implied the need to complete assessments (including practical assessments) and exams, interfering with his time to do research in the lab:

> Inquiry was still there in terms of what you were doing (in the lab) but not something you would necessarily enjoy doing in terms of subject (the background knowledge and 'textbook work' he had to complete to gain his final degree). I am given a problem where I know I will get the result but in research I feel you have to take a topic and search for answers and not be given them. (Sc5/9:11)

The distinction he makes between lab work and constructed problems that are 'just a test of technique' and research emphasises the centrality of D3 in his feelings about behaving like a scientist. Lab work (D2) alone is insufficient, there has to be an element of autonomous control (D3) if the activity is to be real 'research'. When asked to explain this he again identified the notion of 'constructed' work by which he meant activity where the key decisions had already been made:

> Because my research here was all constructed. I was told 'this is the parameter. This is the variable. (Sc5/15:17)
> But there was a time when we were just told, 'this is it. You are just doing this. This is your spectrophotometer. I give you the samples. You're processing this. You're giving me the results. So most probably I was mostly a technician. (Sc5/16:00)

This did not feel, to Sc5, as if it was research. The missing component appeared to be D3 - the chance to engage as an active researcher pursuing his own agenda rather than simply following instructions from others. However, he did claim that his position at university now allowed him some responsibility for driving research projects across his department, more widely shown as a rise in his trace [6]. This was unusual as most interviewees reported a fall in research activity as they were embedded in the day-to-day work of university teaching.

Sc6, Female, Chemist

Sc6's storyline shows a significant jump in her ability to think and behave as a scientist when she took a job in a professional pharmaceutical lab [1]. She explained that at A-level (a 2 year course followed in many UK schools prior to university) [2] she was learning *about* science and mainly D1 whereas in her job she was actually thinking and behaving like a scientist. When asked what 'thinking like a scientist meant' she explained:

> Thinking like a scientist…well, I'm looking at what is in front of me in the lab, thinking about why it's happened, what I need to change to make it happen the way I want it to happen… or what's gone wrong … um any other different ways I could get to the same result… or experienced colleagues that could contribute to the experiment. (Sc6/5:00)

When asked if she had done any practical work in school at A-level she said 'not really' and when further questioned about her A-level lab work experiences she was clearer:

> I wouldn't count that (practical work at A-level) as 'being a scientist' because it's very much following a recipe…knowing what I now know about teaching , it's very much following a

recipe…and that's very different (to science)… science is very much unknown territory. (Sc6/7:42)

Her A-levels emphasised D1, the content required for passing the examination, and D2 in terms of recipe-driven practical work. Only when she reached the professional lab did she experience the range of D2 and any aspect of D3 - a chance to engage with significant others in a task that involved her making a contribution to a real research project.

The Storyline then jumped downwards [3 to 5] which corresponded to changes which reduced her personal control including routine work and teaching and [5] when she took time out to have children:

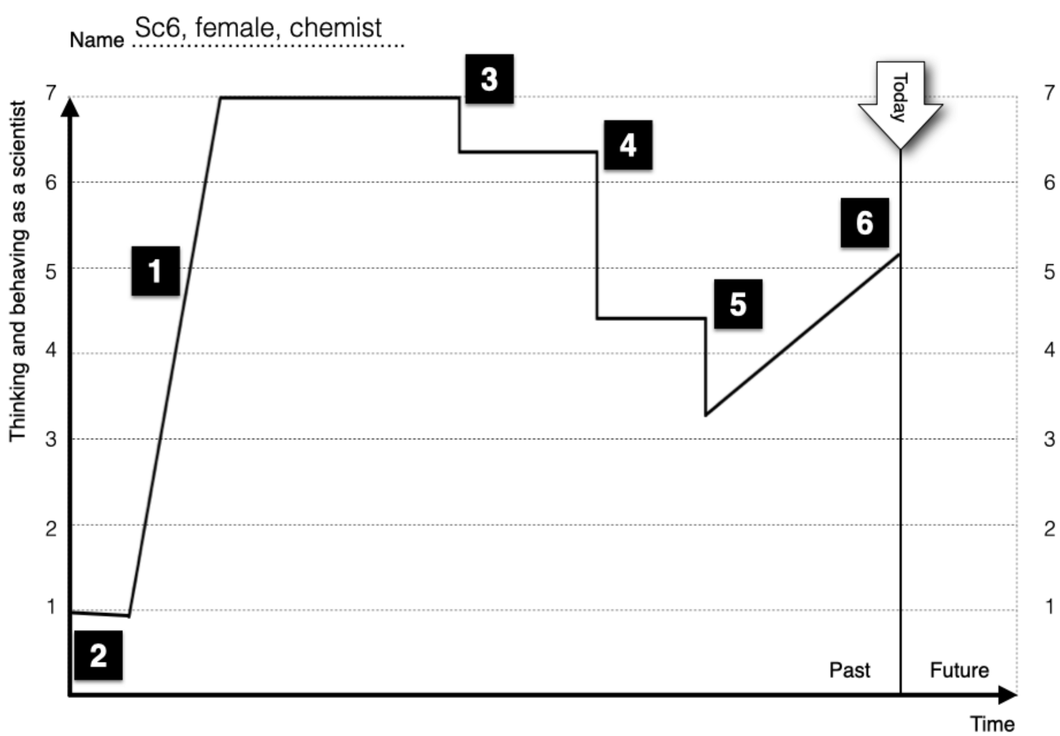
> I was very much on a treadmill of routine analysis and though I did have to use my brain … most days… but sometimes there were times when I just had to tap numbers into computers and things like that …so it drops off there … [3] (Sc7/9:00)
> I got a lectureship … and my priority there became teaching, preparation of teaching, assessment …getting to grips with that type of role and my research, what I call the real science, my research, dropped off to almost one day a week. [4] (Sc6/9:35)

When we asked what she meant by 'real science' she explained:

> Real science is trying to find answers to things broadly…or finding the methods, new methods to find answers to things. (Sc6/10:30)

Figure 8

Sc6 Completed Storyline Form



She went on to explain that she was not really doing any 'real science' - that had been delegated to her students:

> At this point I'm thinking the only science that is happening here is in my … three PhD students. (Sc7/10:45)

Clearly, she feels limited at this time [3 to 5] because her personal opportunity to do 'real science' directly (i.e. conducting the research *herself*) was limited not by D1 (she is at the peak of her field) or D2 (she has contributed to significant laboratory projects) but because her autonomy (D3) has been severely reduced by the time demands of management. However, as she reflected further, she saw herself acting as a 'consulting scientist' by which she meant that she was involved, admittedly at one step removed, from the physical research by offering high level and strategic advice and support to her PhD students [6] playing to her strengths in D1, D2 and D3 explaining the rise in her storyline trace.

Sc7, Female, Physiologist

Sc7 was keen to be a scientist from an early age influenced by a talk given by Heinz Wolfe at her school. Sc7 described her home as being supportive with high expectations. This illustrates a significant feature of D3 in science. To opt to become a scientist, a decision often taken at a relatively young age and largely mandated in the UK prior to age 16 where choices about A-levels can effectively rule out later ambitions to pursue science, involves issues of personal choice (autonomy) that includes emotions, values and a sense of self. In the case of Sc7 the significance of important others (relatedness in SDT terms) was clear. Interestingly, she was not obviously 'good' at science - she volunteers that at school she was only tangentially involved in the practical work central to D2:

> I went to a really awful school [1] where girls weren't really encouraged to do science … I was in a physics class with 5 girls and 35 boys and we weren't allowed to do experiments because we were girls. But sixth-form college was very different, much more academic so that's how I managed to get to university. (Sc7/4:52)

However, when she arrived at university [2] the disappointment was considerable:

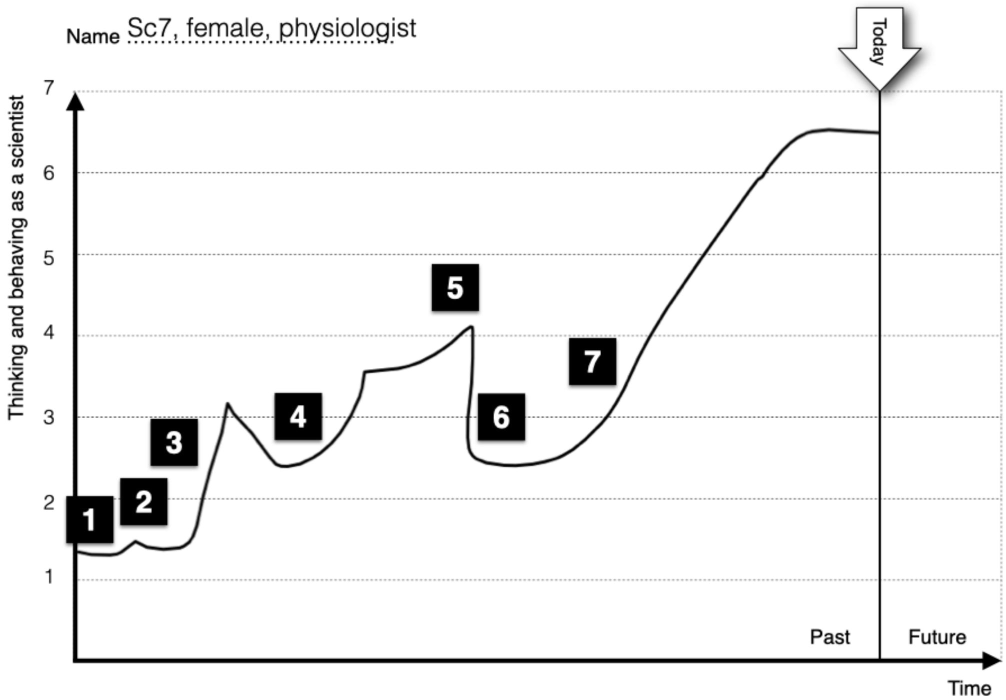
> I found it (the university course) the most boring thing on earth. I did biochemistry and I remember sitting in the first lecture, I was so excited to go to university and it was so boring and I just thought … "Oh no! Why am I here?". (Sc7/5:40)

Again, an event with clear D3 references changed things. She found herself working in a lab [3] and 'doing research' with people who she valued and who valued her:

> I really scuffed my way through university until I got to the final year and then … you had to pitch for a final year project and I applied for the professor's project … so I ended up in the PhD lab and it was fantastic! Doing research and it was amazing! And I just thought "This is what it's all about!" And I went from being bottom of the class to getting a 2:1. (Sc7/6:08)

Figure 9

Sc7 Completed Storyline Form



After graduation she got a job as a part time technician [4] and was later asked if she would like to do a PhD. Her research explored the immunological basis for recurrent miscarriages and eventually produced a significant contribution to knowledge and a step towards treatment for this condition. This was significant and important work that matched her skills and her values showing clear D3 links. At [5] she took some time out to have children. Over subsequent years, a teaching post at a university [6] led to research opportunities [7] and then supervision of PhD students until she was running a department with her own research grants. At this point she was acting as an advisor in a range of research and teaching projects. This fed D3 in terms of the personal significance of the work (she chose to do this) and the value ascribed to her work (D3 competence) by significant others (D3 relatedness):

> The Head of research encouraged me to apply for some funding for a PhD student …so I got that student and she was successful and then I acquired another one … and another one … and then I got a grant … and then because I'd now got people in the lab and because I was becoming more senior I was seen as someone who knew how to do it and so other people invited me to be on their supervisory teams… I now spend most of my time thinking like a scientist … both in my teaching and my research. (Sc7/14:52)

When asked if her teaching interfered with her research she was clear: 'most of my psychological time thinking about research' (Sc7/16:40). She talked about mentoring her PhD students, reviewing papers for a journal and analysing data for an internal project at the university as examples of her 'thinking about research'. She explained that her brain was 'always thinking about analysis really' (Sc7/19:08) which she equated with 'thinking scientifically'. Most D2 references amongst other interviews were to practical work, whereas, in Sc7's case, she was doing no practical but working with others in terms of the aims, variable control strategies, and interpretation of the results rather than handling the equipment. However, when asked to talk about the time, early in her career when she was working in the PhD lab in London [3], and why she felt that was a time of growth in her thinking as a scientist she was clear that it was not simply about control of variables (D2). The references to personal autonomy and the opportunity to make decisions (D3) are very clear:

> It was weird … it's a combination of fun as in, the PhD students were having such a laugh, they were just enjoying doing science so much. They were working really hard … get in really early and go home really late and what they were doing was really complicated, interesting and complicated. Whereas all my experience of lab classes up to then was essentially cooking …. essentially taking ingredients, putting them in a pot, heating them up and seeing what happened and, you, following a protocol is not exciting. (Sc7/20:10)

Sc8, Male, Chemist

Sc8 recognised very clear threshold points related to growth in skills (D2) and personal autonomy (D3) concerning an 'independent research project':

> 'There are very definitely threshold points … some of the more significant ones were at the start of my degree [1] … I did a four year integrated Masters program, I think the first two years were very different to the second two years and so Year 3 [2] was a big step up in scientific methodology because we got to do an independent research project during the third year of my degree. That made a significant difference. In the fourth year [3] we did a larger scale independent project so again that contributed largely to that development which was built on during the PhD. (Sc8/2:58)

Returning to his years at A-level [4] he complained that they were stressful and did not contribute to his developing as a scientist because of assessment pressures:

> A-level was mostly targeted at exam performance, so there wasn't much scope for designing experiments, testing hypotheses…seemed to be a lot of practising for the exam questions. (Sc8/5:08)

He found that at degree level there were changes, in both educational philosophy and educator desires, towards encouraging students to develop their own ideas (D3) and 'the thought processes of a professional scientist' (Sc8/5:58). When asked what these 'thought processes' are, Sc8 was very clear and provided a summary of the typical scientific method that sits largely in D2 but with references to theories from D1:

> So, it is based on a cycle largely making an observation of a phenomenon in the real world, developing an idea or hypothesis that rationalises why that observation can be made in the way it is … then design an experiment to test that hypothesis… evaluating the results. (Sc8/6:05)

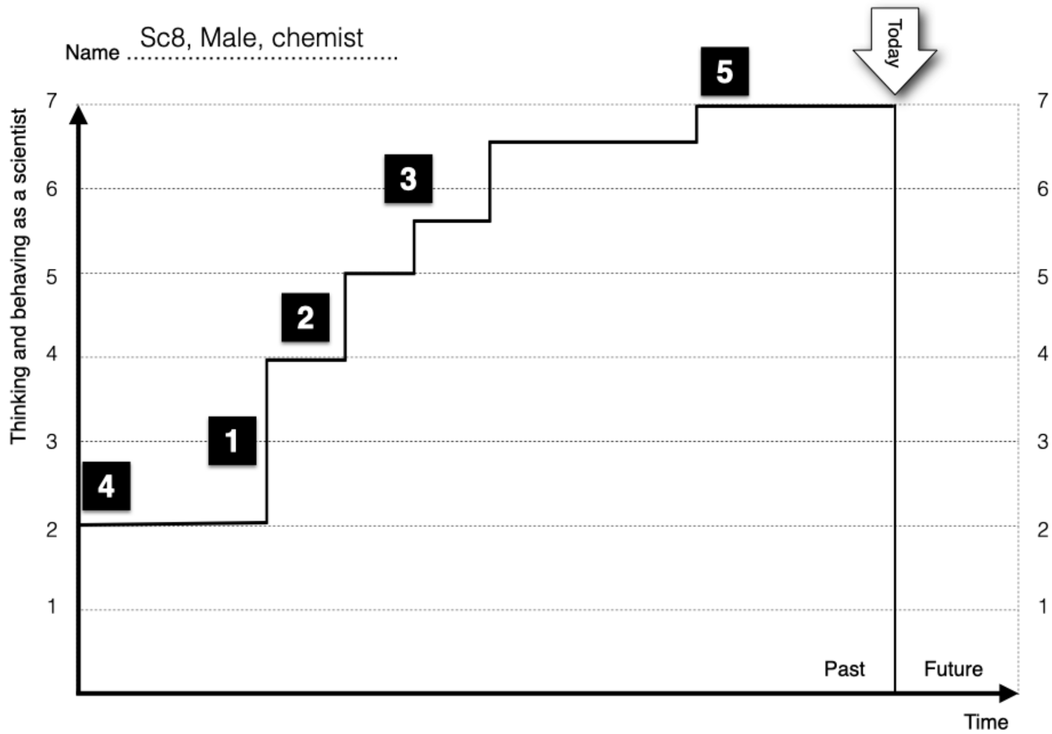
Despite his description of 'the scientific method' (D2) as a simple, almost personal process he emphasised collaborative working in teams as a key part of being a scientist:

> I think it (team work) is an important part of science. If you look at any of the major research challenges they are all interdisciplinary in nature. In order to form a productive

> research team you need contributions that span the conventional discipline boundaries … it requires that close level of collaboration. (Sc8/8:03)

Figure 10

Sc8 Completed Storyline Form



Sc8 also distinguished between training in formal skills and independent research:

> A lot of the formal training that was put into place in the first year and the second year of the degree helped bring out those skills (research skills involving scientific method and collaboration) but I really think it was going into the lab and putting a lot of that into practice in independent research that brought it up to almost where it is now. (Sc8/9:45)

The conversation then moved on to discuss the Problem-Based Learning (PBL) curriculum used at Sc8's university. When it was introduced he identified lack of student engagement with their science courses as a significant issue - almost a nervousness about even discussing chemistry with their peers and tutors in case they displayed a lack of understanding in their answers. When asked how they solved this problem he did not reach for better teaching on scientific method (D2) but in a shift towards supporting student autonomy (D3):

> The only way we've been able to solve that problem is to give students almost complete control of that type (PBL) of learning experience. (Sc8/18:44)

His storyline trace finished at the maximum indicating he was thinking and behaving as a scientist more now than at any other time in his career [5]. This was unusual as most of the storylines tend to dip slightly as participants took on more teaching or administration duties. Sc8's rise seemed to be linked to his active involvement in the research his students were initiating and doing, a component of the department's PBL approach. When asked if there was anything he would like to add about his experience of becoming a scientist he was very clear - it was about personal control and the degree of autonomy available (D3):

> I think it's entirely down to being put in control of situations to develop as a scientist is to be given that responsibility … plan, design, carry out and reflect on your own experiments … the overall message is that it's got to be something that you're in control of, something that isn't scripted, something that there isn't a fixed end point to … unlike some of the early level educational experiences people have.' (Sc8/23:44).

Discussion

Given the obsession with D1 and D2 in the research literature about science education we anticipated that the storylines would have shown a gradual rise as participants mastered more of the theoretical background (D1) and gained more skills (D2). However, the emphasis for the eight participants seems to be much more around D3, and high levels of D3 always seemed linked with high levels of scientific activity, while low levels of D3 always indicated a lack of what the participants called 'real science'. This corresponds with our contention that D3 includes 'psychological energy' (Bevins & Price, 2016) which is an essential factor in driving scientific activity. Psychological energy, as we conceive of it, is produced by a combination of autonomy, competence and relatedness in the same way as intrinsic motivation as described by SDT. (Deci & Ryan , 2006; 2012). Our 3D model moves scientific activity from a process to be completed by implementing aspects of prior knowledge (D1), alongside relevant skills (D2), into a conscious strategy adopted by an autonomous individual (D3) using aspects of D1 and D2 to achieve a personally valuable goal.

References to autonomy and being 'in control' appear regularly (Sc2; Sc5; Sc6) and are always related to high points in participants' sense of 'doing real science'. This autonomy brought responsibilities, participants talked about working harder, seeking to understand issues more deeply and 'feeling bad' when things did not work out (Sc1; Sc2). Participants also expressed the opposite perception - that the lack of autonomy reduced the activity to meaningless techniques or procedures (Sc4; Sc5; Sc8). The participants made a number of comments about increased competence with some identifying teachers (Sc5) or changes in courses (Sc2; Sc7) as significant. Participants distinguished between simple 'rule-following' and 'thinking like a scientist' implying that their sense of competence was more deeply-seated than being simply a high mark in an assessment (Sc3; Sc6; Sc7). Reviewing the conversations, there are many references both to the notion of relatedness concerning being part of a scientific team or even group of friends (Sc4, Sc6) with shared interests and capabilities, a relationship with a particular teacher or mentor and even, in some instances, a relatedness linked to a sense of the planet and the natural (Sc3; Sc5), or family and friends (Sc6; Sc7). Sc8 simply claimed that scientific research is inevitably collaborative.

Reviewing the 3D Model

We argue that the conversations and our analysis supports our claim that science is best thought of as an activity that has three related but independent dimensions. Our second claim is that a deficit in any of these dimensions will lead to activity that is not perceived as 'behaving like a scientist'. If D1 or D2 are weak the activity is not 'science' since it does not draw on scientific domain knowledge or scientific method. It may be a valuable activity but it is, by definition, not science. Our data shows that a deficit in D3 has a more subtle effect. People using scientific ideas and employing scientific method can be engaged in what some of our interviewees (Sc2; Sc8) described as, 'training for science' where an absence of D3 meant authentic science, which they often refer to as 'real science', was not happening - both in terms of their comments and the, sometimes dramatic, falls in their Storyline traces. In some instances a weakness in D3 led to the scientist absenting themselves (in the case of Sc7 to travel the country with a student rock band!) whereas in others it led simply to continuing to work hard, but with little sense of purpose or achievement (Sc3, Sc4). Arguably this second, larger group of 'willing conscripts' were still motivated sufficiently to continue their 'science' courses and activities but they felt it was not authentic. The motivation had been lost but something more significant to the discipline had also been lost. The message is repeated in other conversations: science without strong D3 is not just boring or un-motivating, it is not authentic science in a very significant sense: it is not experienced as the 'genuine article' by the people engaged in it. So, while D3 shares some commonalities with motivation (whether for science research or studying history or practising skateboarding) it is not exactly the same.

Implications for Science Education

We believe that we have provided evidence to support the idea that D3 is a critical part of scientific activity and suggest that this has serious implications for science education and the relative amounts of time spent on different activities within it. When we explore approaches to teaching science we notice that much of it, particularly in England, focuses on gathering more knowledge (D1) and practising routines and skills described as 'scientific method' (D2) in preparation for high-stakes assessments. Further research from the US and Australia report the curriculum cramping effect of heavy assessment instruments (Jones, 2007; Polesel et al., 2013). Even ignoring D3 for a moment, teachers in the UK regularly report the pressures on them to deliver large amounts of material in a limited time and that this prevents them from doing investigative work (a possible incarnation of D2) outside the limiting demands of the assessment regimen (Bevins et al., 2019). Having spoken to teachers from both the US and India, it seems that many of them share a similar perception that the science curriculum is already content heavy and the assessment regimen is dominated by D1.

But if we argue that D3 is an essential part of scientific activity surely there must be some shadow of it in the curriculum? In our study of the National Curriculum Science Orders for England (Department for Education, 2013) we can find no overt references to the components we anticipate fitting into D3 (autonomy, relatedness, competence) and even the references to the nature, processes and methods of science refers to it as 'working scientifically', reducing it to a sort of *cognitive mechanism* with no reference at all to society more widely until students reach the age of 14. While versions of the NoS from the US (see Table 1 earlier in this article) may contain some connection to some aspects of our D3 (the notion of 'a scientific community' and that 'society and culture influence each other', has shadows of D3's references to relatedness), any references to a scientist as a person rather than as a kind of biological and cognitive mechanism following a set of shared rules to produce an agreed understanding of a topic are not front and centre.

Similarly, problem-based learning and socio-scientific approaches can provide opportunities for the 'scientist as a person' with a social context to be revealed and communicated to students in lessons. However, these approaches are still relatively rare, despite the fact that a range of meta-analyses (Minner et al., 2010; Schroeder et al., 2007; Schwichow et al., 2016) show that emphasising the real world context of the science and allowing collaborative working not only increases motivation but improves performance. Indeed, it may be that the enlivening of otherwise boring material by an exciting or dramatic context may be the sole motivation for including aspects of D3 in the teaching and learning strategies rather than to reflect the nature of scientific activity itself. However if D3 is present as an integral part of scientific research, as claimed by the researchers quoted in this article, then supporting it is not simply a possible teaching strategy or a way to make material about electricity 'sexier' but a fundamental requirement?

Possible Ways Forward

The researchers and educators quoted in this paper are amongst the most successful of their years, they are the people who make it through to research and teaching posts, and they are clear on two issues. The first is that no D3 means no 'real research' (Sc8) with only 'constructed' (Sc6) problems on offer. The second is that their science education did not always provide the necessary third dimension in their studies and, in some instances, almost drove them from a career in science. If we are to help students to develop into 'real' scientists we have to accept that D3 is not a desirable extra, any more than D1 or D2 is, but an essential requirement. To avoid D3 or maintain that it is less worthy of a place in our, admittedly crowded, curricula may mean that most students have a limited experience of 'real science' and are unlikely to become the research scientists and technologists we need to solve some of the global problems we now confront as a species.

But how can one 'teach' students 'autonomy', 'competence' or 'relatedness'? These are not simply facts and theories or skills and capabilities and cannot be 'taught'. Maybe they are 'caught' by students as they work in classrooms that support student autonomy, that allow working in collaborative groups and aim for mastery rather than the performative goals of traditional public examinations? Researchers working in SDT have been looking at environments that support, or reduce, student autonomy and other related D3 factors for a number of years (Hyungshim et al. 2016) and have published useful advice on these matters.

In our previous paper (Bevins & Price, 2016) we suggested D3 existed. In this paper we present evidence that it is familiar to scientists who have actually engaged in research. We now propose to create D3-friendly science materials and approaches and evaluate their impact in terms of student motivation, perception of the nature of science, and eventual achievement.

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Broadening Conceptions of STEM Learning: "STEM Smart Skills" and School-Based Multilingual Family Engagement

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ABSTRACT

STEM education researchers are well aware of the need for increased access and inclusivity in Science, Technology, Engineering, and Mathematics (STEM) education for students from culturally and linguistically diverse (CLD) backgrounds. One of the many barriers for students from underserved cultural and linguistic groups is the difficulty of connecting families to school models of STEM education. This is one reason we advocate for improvement in culturally relevant STEM curriculum and content instruction. This commentary does not focus on STEM content instruction, although we certainly believe children from CLD communities deserve high expectations and high quality, culturally sustaining, STEM pedagogy. In this article we discuss non-curricular skills that are vital to success in STEM – and the advantages of sharing with family members the importance of particular essential life skills that support STEM learning. Communicating these essential "STEM Smart skills" showcases the power and influence that families have in kids' STEM learning. In this commentary we describe a school-based family STEM night that included a demonstration that success in a STEM task is not based primarily on content knowledge but on "STEM Smart skills." Many family members found success in the activity, regardless of parents' educational level or background in STEM. Family members' rich life experiences, critical thinking skills, and cultural knowledge include these "STEM Smart skills." We argue that teachers and schools should communicate to families about these life skills. This focus can benefit students by highlighting family members' power and role in teaching and modeling, essential skills for students' STEM success. This focus also can benefit educators by challenging common stereotypes about families from underrepresented cultural and linguistic backgrounds. In this way, acknowledgement of "STEM Smart" life skills could play a small part in dismantling structural racism and inequitable power relations between schools and communities.

Keywords: STEM education, STEM Smart skills, family engagement, culturally and linguistically diverse (CLD) families, culturally relevant pedagogy

Introduction

Science, technology, engineering, and mathematics (STEM) education research identifies STEM education and STEM career paths must become more accessible and inclusive for students from *culturally and linguistically diverse* (CLD) backgrounds (Jong et al., 2020). Previous research has documented a number of structural barriers in education and society toward STEM achievement among students from underrepresented groups (Buck et al., 2020; McGee, 2020). This article focuses on one specific aspect of increasing equity in STEM education – strengthening the alliance between schools and families in culturally relevant and culturally sustaining ways. One way to increase CLD students' interest in pursuing advanced STEM education is through communicating and showcasing STEM strengths and connections that already exist within cultural communities (Johnson et al., 2014; Magee et al., 2020). We suggest one approach to family engagement in STEM that focuses not only on STEM-specific topics, but also on non-STEM-specific (and oftentimes non-academic) life skills that are essential for success in STEM education. Communicating these essential "STEM Smart skills" showcases the power and influence that families have in kids' STEM learning.

In this commentary we describe a school-based family STEM night that demonstrated success in a STEM task that is not based primarily on content knowledge but on "STEM Smart skills." We found many family members had success in the activity, regardless of parents' educational level or background in STEM (Hoffman et al., 2021b). Parents' and guardians' rich life experiences, critical thinking skills, and cultural knowledge include critical "STEM Smart skills." We argue that teachers and schools should consider these "STEM Smart skills" and communicate about them to families. This focus benefits families by highlighting parents' and guardians' power and role in teaching and modeling essential skills for students' STEM success. This focus also benefits educators by challenging common stereotypes about families from underrepresented cultural and linguistic backgrounds. In this way, acknowledgement of "STEM Smart skills" plays a small part in dismantling structural racism and inequitable power relations between schools and communities.

Culturally Sustaining Pedagogy: A Conceptual Framework

We propose a model of STEM family engagement based in culturally sustaining pedagogy (Paris, 2012; Paris & Alim, 2017). This asset-based pedagogy builds from heritage and contemporary practices of communities of color while critically examining assumptions about the (lack of) value of community practices versus dominant cultural practices. Like other asset-based pedagogies, culturally sustaining pedagogy views communities of color as important sources of language/literacy practices and cultural ways of being that support students' academic achievement. Simultaneously, culturally sustaining pedagogy seeks to *sustain* "linguistic, literate, and cultural pluralism as part of the demogractic project of schooling" (Paris & Alim, 2014, p. 88). Scholars have applied culturally sustaining practices in mathematics (Leonard, 2018) and science (Oatman, 2015) learning contexts. In particular, we argue that family engagement activities can amplify and increase the visibility of the ways in which communities' rich cultural knowledge and life experiences are relevant to STEM learning.

STEM Education and Family Engagement

Research over the last decade consistently shows a variety of barriers to STEM education and career accessibility for students from underrepresented cultural and linguistic groups. These barriers include educational quality in many non-White and low-income communities, access to opportunities to apply STEM skills, lack of role models and mentorship in STEM careers, lack of culturally relevant pedagogy in K-12 classrooms, school and work environments that stereotype or devalue students' identities, and marginalization in the workplace (Jong et al., 2020; McGee & Robinson, 2020). We believe a majority of educators want to connect with families as allies and advocates for their children's educational achievement. Yet many educators – most of whom are from the White, monolingual English-speaking background dominant in U.S. schooling – are only comfortable engaging parents and STEM content through the dominant STEM curricula that reflect a White, Eurocentric cultural framework (Mensah & Jackson, 2018; Leonard et al., 2010). Likewise, it can be difficult for families to connect to their children's schooling if they do not feel connected to the school due to cultural or linguistic differences or if they do not have much formal education themselves (Thomas et al., 2020).

We believe that all parents need to know that success in STEM comes not only from disciplinary or content-based knowledge but also from particular essential life skills that support STEM learning. Communicating these essential "STEM Smart skills" to CLD families can be especially significant in acknowledging the influence and importance of families' cultural heritage, funds of knowledge, and professional skills (Gonzalez et al., 2006; McKenna & Millen, 2013). Understanding how caregivers can support children's mindset, tenacity, and critical thinking skill development helps them realize the power and influence that they have over their students' STEM learning. Showcasing families' "STEM Smart skills" also is instructive to educators and administrators who may be accustomed to viewing their students' families through a too-common deficit-based lens (Hoffman et al., 2021b).

We want to emphasize that we are not focusing on family engagement initiatives because of incorrect perceptions of families of students as a barrier or impediment to students' STEM learning. On the contrary, we agree with current family engagement research that challenges such traditional top-down (and often deficit-based) "parent outreach" initiatives (Albrecht, 2020; Goodall & Montgomery, 2014). Instead we aim for a cooperative asset-based approach that focuses on families' funds of knowledge and STEM-related life skills. In this vein, we approach our role in school-based family engagement activities not as the visiting experts, but as facilitators with the opportunity to point out to both parents and school staff the connections between parents' prior knowledge, families' cultural heritage, and the skills students need for success in STEM fields. In this commentary, we describe a school-based event for families where we demonstrate that success in a STEM task is not always based on content knowledge.

We argue that teachers and schools should consider explicitly addressing "STEM Smart skills" both with students and with families. These "STEM Smart skills" are life skills that caregivers can support at home. When parents and guardians realize their own power and role in teaching "STEM Smart skills," they recognize how essential they are in students' STEM success. We urge STEM educators to consider the usefulness of communicating the importance of these "STEM Smart skills" to families. We encourage STEM education researchers to consider further research into the role of these "STEM Smart skills," both in STEM learning and in STEM family engagement.

The Need for STEM Smart Skills

Educators and families alike are nurturing young STEM thinkers who will solve the problems of today and develop new tools to resolve future problems not yet encountered. The upcoming generation must be prepared to address: (1) societal needs for new technological and scientific advances; (2) economic needs for national security; and (3) personal needs to become fulfilled, productive, knowledgeable citizens (Zollman, 2012). From an equity-oriented approach, the need for increased skills relates to overlooked needs of learners from marginalized communities and the overdue need for social justice in STEM education (Barton, 2003; Leonard et al., 2010). Robert Berry III, Past President of the National Council of Teachers of Mathematics, along with his co-authors (2020) state that teaching mathematics for social justice is critical for four reasons: building an informed society; connecting mathematics with students' cultural and community histories;

empowering student to confront and solve real-world challenges they face, and helping students learn to use mathematics as a tool for social change (Berry et al.).

As we said before, preparation for STEM innovation requires more than just content knowledge or exposure to STEM content. Families can buy all the STEM-marketed kits and toys they want, but these predesigned, partially assembled kits will not prepare young "STEM Smart" citizens for meeting upcoming challenges. An intellectual risk taker's mindset, an innovator's tenacity, and a skeptic's critical thinking skills are must-have attributes all children will need to solve the problems of the future. Children need to develop perseverance and critical thinking to analyze multiple arguments, to innovate possible solutions, and to advocate for causes they support.

Identifying Five "STEM Smart Skills"

When engaging with families, we stress five "STEM Smart skills" for students. With families we use the acronym **SMART** as a mnemonic device to help us communicate these essential life skills for STEM learning: **S** for productive **s**truggle, **M** for usefulness of **m**istakes, **A** for STEM's relevance for **a**ll people, **R** for intellectual **r**isk taking, and **T** for critical and divergent **t**hinking.

1. "Struggle can be productive" (S): We feel that the importance of working through challenges is important to "STEM Smart" thinking. When we talk about "struggle" in terms of learning, we're specifically talking about persistence through tackling tough problems. A student with a "STEM Smart" mindset is willing to tackle tough concepts and problems that do not have instant, easy answers. Key components to this type of learning are persistence and reflection about what works and what does not work. In "productive struggle" (NCTM, 2014), it is vital that a student's efforts are *productive* so as to reinforce a student's self confidence and willingness to persist doing challenging tasks.

We discuss with families the importance of giving students time and opportunity to manage their struggles through adversity and failure by not stepping in too soon or helping too much. When adults step in too quickly to solve a problem for students, they take the intellectual work away from the learners (Warshauer, 2015). Hiebert and Wearn (1993) and Borasi (1996) found that this practice repeated over time can contribute to students viewing struggles with learning mathematics negatively instead of viewing struggle as an opportunity to learn. In our experience, parents and guardians appreciate hearing about this relevant research.

2. "Mistakes are how we learn" (M): STEM skill development cannot flourish without acceptance of mistakes as natural, even welcome, parts of the learning process. Being wrong makes us uncomfortable, but students cannot develop and discover without mistakes. Human ingenuity and invention is inextricably connected with making mistakes. Asking parents what notable "mistakes" we value in our culture may yield answers from champagne to Coca-Cola, from popsicles to penicillin, from sticky notes to Silly Putty, and from rubber to Velcro.

Many students feel that their work needs to be perfect to be worthwhile. Perfectionism can be dangerous, as it has been linked to anxiety disorders and other forms of psychological distress. Even in small doses, this rigid unwillingness and fear of making mistakes prevents children from accessing a powerful tool for learning. Research on math anxiety, and now STEM anxiety, for the past 45 years has shown that math anxiety is a *taught* negative mental and physical response. Researcher Sheila Tobias (1993) said math anxiety is a mental phobia that affects children's motivation, self-confidence, attitudes, beliefs, and thus achievement. In our experience, parents and guardians are interested in concrete suggestions to address students' math anxiety; this relates to other areas of STEM as well in a broader discussion on the important role of making mistakes.

3. "STEM is for all people in all places" (A): Dominant popular culture and school curriculum alike reflect a history of not recognizing or valuing certain groups of people and certain types of knowledge in STEM. Since adults grow up surrounded by dominant cultural values, we often do not realize the messages children are receiving.

Students need to know that everybody has a STEM heritage. Many students (of all racial, cultural, and linguistic backgrounds) are unaware that people throughout time and across the world have made discoveries and developed technologies not taught in U.S. schools. Often school experiences focus on European and North American inventions – particularly those inventions made by White males for application in profitable industries. As one example, most Americans were taught that Greeks were pioneers of science and mathematics, when actually the Aztec, Incan, Nubian, Malian, Congolese, South African, Kenyan, Egyptian, Indian, and Chinese civilizations all utilized mathematics and astronomy in their cultures much earlier (Prescod-Weinstein, 2015).

It is critical for students to understand that STEM is for everyone for at least three reasons. First, students need to break the cycle of stereotyping their peers' STEM potential based on racial and gender stereotypes (Jong et al., 2020). Second, STEM professional and educational spaces need to become more welcoming to students from underrepresented groups (Leonard et al., 2010). Third, we want to combat the too-common imposter concerns among CLD students that perhaps STEM fields are not for them (Boaler & Greeno, 2000; McGee, 2020). Actively modeling a broader cultural and racial view of STEM – as well as dispelling stereotypes surrounding computer geeks and lab coats – provides our kids with an equitable vision for STEM and thus a stronger "STEM Smart" foundation.

4. "Reward intellectual risk taking" (R)**:** Children's intellectual risk taking is based on their natural sense of wonder and curiosity about the world and the way things work. Children who are willing to take risks develop a tendency to be open-minded, to generate multiple options, to explore alternative views, and to have an alertness to narrow thinking (Grotzer, 1997). Children's disposition toward wondering, problem finding, and investigating relates positively to an adventurous mindset.

As educators we need to foster an environment that allows children to go beyond their comfort zone. "STEM Smart" kids need to be bold. Whether it is learning a new skill, creating a business, or searching for a cure to a pandemic, having the bravery to take risks is an essential "STEM Smart skill."

5. "Think before you trust" (T): Some technical skills taught in today's STEM courses will be obsolete by the time our students are adults. But "STEM Smart skills" are never obsolete. They are the habits of mind that give our children the agility to apply their existing knowledge and skills to new contexts. This final skill in our acronym alerts parents and guardians to the importance of critical thinking – itself a set of skills that is becoming increasingly important in the age of digital literacy and social media.

We explain critical thinking as a combination of several intellectual processes. Critical thinking involves deciding what knowledge is relevant to a situation, evaluating information for quality, and applying the relevant knowledge to make informed decisions. "STEM Smart" critical thinking also includes questioning others' thinking, recognizing contradictions and biases, and admitting flaws in our own thinking.

In the current U.S. cultural climate, many people voice concern about finding and evaluating trustworthy sources of information (Ortutay & Klepper, 2020). Teachers at every level from elementary to graduate school have voiced concerns about young people's information literacy and media literacy skills. The ability to evaluate information, to make judgments, and to think critically, is key to a successful STEM mindset.

Lack of skill and judgment in this key area of STEM thinking has a wide-reaching influence on our society in areas ranging from ill-considered government policies to significant numbers of Americans refusing to believe highly qualified scientific experts (Hayhoe & Schwartz, 2017). The Pew Research Center reported most Americans believe that science has benefited society, but fewer than one-third of Americans trust medical research scientists to give fair and accurate information (Funk et al., 2020). Consumers of media of all kinds, print and digital, need basic skills in questioning reported trends, interpreting statistics, identifying bias, and recognizing the validity and reliability of data.

Current K-12 students often view a friend's reposted quote on social media as equally or more valid than an article published in a research journal. We want kids to question: the sources of their information, the possible bias of the sources, the resources these sources use for the information, and the analysis that was conducted on the information.

Considerations for STEM Family Engagement Activities

Current literature supports many possible forms of family engagement (Baker et al., 2016; Mahmood, 2013). As one example, we have done several "STEM Family Night" events hosted by elementary schools with large numbers of Spanish-speaking Latinx families. We advocate approaching such STEM family engagement activities with five key considerations:

- Center the Event in Existing Community Relationships (Albrecht, 2020);
- Connect with Community Knowledge, Heritage, and Values (Magee et al., 2020);
- Choose a High-Interest, Integrated STEM Exploration Activity (Suh et al., 2020);
- Make the Activity Hands-On and Challenging (NCTM, 2014); and
- Focus on STEM as Inquiry for *All* Participants (Hoffman et al., 2021b).

First, in terms of *existing community relations*, research identifies relationships with students as keys to learning families' "funds of knowledge" and finding natural community partners (Gonzalez et al., 2006; Moll et al., 1992; Rios-Aguilar et al., 2011). These funds of knowledge may be outside of commonly "aspirational" STEM fields such as engineering or medicine. Most jobs and learned skills require some level of expertise that can be related to "STEM Smart skills."

Second, both culturally relevant STEM pedagogy and current family engagement research emphasize the importance of *connecting with community knowledge, heritage, and values* (Magee et al., 2020; Thomas et al., 2020). Choose a focus and activity that centers the experiences and identities of the families who will be attending (Kayumova et al., 2015). Recognition of the knowledge and resources families possess and bring into the school is at the heart of culturally sustaining pedagogy (Paris, 2012).

Third, effective STEM exploration activities for family engagement are *open-ended activities* that encourage *hands-on problem-solving*. As family engagement events usually occur outside of the regular academic day, it is easier to do an integrated approach to STEM than in a traditional school curriculum. Further, open-ended activities lessen the impulse to "find correct answers" or "teach parents" some STEM content.

Fourth, the activity for the event should be *novel, challenging, and interactive*. It should require physical activity yet be accessible to multiple ages and abilities. Examples could include a competitive challenge of building the tallest freestanding tower out of dried spaghetti noodles, masking tape, string, and marshmallows. Another small-group challenge is building a "ringlet" arch using only Pringles potato chips.

Fifth, we want *all family members to view themselves as learners*. Our activities demonstrate that fluency in English or possessing specific content knowledge is not necessary to STEM learning. At the end of the activity, we ask participants to join in a reflection to discuss what STEM is and what STEM is not. STEM skills are not content knowledge, but STEM success does not require the ability to use knowledge in solving problems. We want parents to encourage their children to become intellectual risk takers with the tenacity to tackle tough problems and the critical thinking skills to separate scientific information from opinions.

One Example of a STEM Family Engagement Event

One successful event we have done at elementary schools is “STEM Family Night.” These events center around an interactive activity. We conducted two such events on weeknights at two different elementary schools in the same school district. At one school, the event was designed for families with children receiving services as English language learners. Parents and guardians were invited via email, and flyers printed in both Spanish and English were sent home in children’s backpacks. At the other school, where a large percentage of students come from bilingual families, the entire student body was invited to the event via bilingual flyers sent home in backpacks as well as weekly email newsletters. (Spanish was the most common language spoken by English learners at both schools, by far, although other languages were also represented.) Invitations for events at both schools welcomed entire families, including siblings.

At both schools, administrators, including the principal, and teachers attended the event and the school provided dinner to all participants. Once families had time to eat and socialize, we welcomed everyone and invited them to move into groups to participate in a marshmallow tower-building group challenge activity.

In hands-on challenges like these, we ask participants to move into working groups having parents sit with other parents, kids sit with other kids of various ages, teachers sit with teachers, and school administrators sit with other administrators. We prefer this grouping strategy because we have found that some parents defer to teachers or school administrators if all adults are grouped together. When parents or teachers are grouped with children, the adults tend to direct the children. When parents are placed at a table with other parents, however, they feel less self-conscious about making mistakes and more likely to take risks and enjoy their errors. (As an aside, we have also noticed that the children relish competing with adults.) When participants represent several language backgrounds, we deliberately mix participants from different languages to show that learning can be accomplished with limited verbal communication.

In our experiences, we observe that parent groups are much more reserved, often needing a lot of encouragement to try divergent ideas. In contrast, the groups of children are eager to experiment with various strategies to tackle the challenge, regardless of whether adults think such strategies might work. At one event, one group of children taped dried spaghetti noodles end-to-end before putting their marshmallow on top. Of course their tower would not stand; it bent over in an arc instead. But that did not bother the kids. They realized that they could make another arc at a right angle to support the first one. The two intersecting arcs supported the marshmallow. That group’s initial mistake produced a better result.

Experiences like these demonstrate the importance of “STEM Smart Skills” and provide openings for conversations about the value of each of the five STEM Smart skills. Participants in the STEM challenge had to be willing to persist through struggle (“S”), make mistakes (“M”), consider the ideas of all group members (“A”), take risks (“R”), and think critically about possible solutions (“T”).

As a physical “takeaway,” we give a bilingual handout with advice for reducing STEM anxiety and supporting a positive mathematical mindset in adults learning with their children (Boaler, 2015; Suh et al., 2021). In our event evaluations, parents described STEM education as more hands-on, enjoyable, and problem-based than expected. They saw the value of communicating in a team, allowing mistakes, and persevering as important aspects of learning "STEM Smart skills" (Zollman et al., 2020).

Findings from STEM Family Engagement Activities

As this is a commentary and not a research report, this article focuses on connecting multilingual family engagement with “STEM Smart skills” rather than sharing empirical results.

However, we do believe that we have learned six important lessons from our work on family engagement:

1. A STEM activity does not need to focus on academic content;
2. The STEM activity focus should begin with families' funds of knowledge;
3. Activities can focus explicitly on "STEM Smart skills;"
4. School teachers and administrators need to take part, but not lead, in the activities to have the opportunity to adjust and expand their views of students' families as partners in education;
5. Family members are excited to be asked to join in STEM education; and
6. Family members appreciate concrete examples of "STEM Smart" concepts and skills.

Implications for Future Research

Based on our experiences facilitating STEM family engagement events, we suggest three main areas for future research. These include the effects of family engagement programming, the effects of highlighting cultural connections, and the effects of "STEM Smart skills" on academic learning.

Effect of Engagement Efforts on Students, Families/Communities, and Educators

First, we encourage researchers to explore the efficacy of STEM family engagement events in developing STEM content knowledge, skills, and motivation. Researchers can also explore the impact of STEM family engagement events on school administrators and teachers – how witnessing family-based informal and collaborative STEM exploration might reduce educators' deficit perspectives of families' interest in STEM, or their ability to support children's STEM learning. Here are some additional questions researchers may want to explore:

- Do families/communities change their view of STEM from being a product to being a process through family engagement activities?
- Do families/communities see students' STEM abilities as fixed or open to growth?
- Do family engagement efforts affect families' perception of their role in teaching "STEM Smart skills" such as perseverance, intellectual risk-taking, acceptance of mistakes, appreciation of STEM in all fields, and critical thinking?
- How do family/community engagement activities increase educators' ability to make meaningful and authentic connections between STEM content and students' lived experiences?

Connections Between Student Cultures, STEM Heritage, and "STEM Smart skills"

STEM education researchers can draw on STEM equity literature and family engagement literature to connect the STEM heritage and funds of knowledge of CLD families with STEM curriculum and instruction in schools. Such research is needed to explore how family engagement can support current scholarly efforts to "decolonize" STEM curricula (Anthony-Stevens & Matsaw, 2019; Howard & Kern, 2019; Kimbrough, 2017; Kimmerer, 2013; Nhemachena et al., 2020; Prescod-Weinstein, 2015). Again, here are possible questions to study:

- Do culturally relevant STEM activities deepen students' and families' understanding of STEM as part of their own heritage, instead of "belonging to" the dominant culture of U.S. education?
- How does family/community engagement support efforts to relocate STEM knowledge outside of traditional narratives of Western scientific discovery?

- How does the use of cultural connections in family engagement activities affect families' view of STEM learning and their children's potential future in STEM careers?
- What effect does highlighting cultural connections in family engagement activities have upon educators' views of their students, their students' communities, and of the intercultural connections possible in STEM teaching?
- What effect do efforts to decolonize STEM curriculum and pedagogy have upon teachers' attitudes and student achievement?

Effect of "STEM Smart skills" on Academic Achievement in STEM

Research also can examine the connection between students' STEM learning and the skills and knowledge we call "STEM Smart skills." Possible research questions here could include:

- What effect does each of the five "STEM Smart skills" have on student achievement in STEM academic areas?
- What are effective ways schools and teachers can increase students' "STEM Smart skills"?
- What are effective ways educators can communicate to families their critical role in nurturing students' "STEM Smart skills" development?

In closing, we wish to emphasize the importance of basing future research efforts in authentic school-family partnerships (or school-family-university partnerships), which centralize the family's role and knowledge in supporting students' STEM learning. Like Moll et al. (1992), we view these "strategic connections" to families as essential to centering our understanding of funds of knowledge around the family's values and ways of knowing rather than the researchers' values and ways of knowing (p. 132).

We view the families' role in nurturing "STEM Smart skill" development as essential to support students' STEM education. Like Moll et al. (1992), we urge an approach to research that values parents' funds of knowledge rather than a traditional approach that centers around our preconceived values and assumptions. A research agenda that integrates the views and values of CLD families aligns with culturally relevant and sustaining teaching practices (Ladson-Billings, 1995; Paris & Alim, 2014, 2017).

The field of STEM education is evolving. As part of this evolution, we strive to be more culturally responsive STEM educators and researchers. The purpose of this commentary is to encourage further discussion about research in connecting the three growing areas of STEM Smart skill development, culturally relevant STEM instruction, and family engagement. Such a focus can benefit students by highlighting family members' powers and roles in teaching and modeling essential skills for students' STEM success. This focus also can benefit educators by challenging common stereotypes about families from underrepresented cultural and linguistic backgrounds.

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