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Quantitative and Covariational Reasoning in Calculus: Exploring Students' Thinking About Functions and Derivatives

Thembinkosi P. Mkhatshwa
Miami University

ABSTRACT

This qualitative study investigated calculus students' quantitative reasoning and covariational reasoning about functions that are situated in real-world non-kinematics contexts. Specifically, the study examined students' thinking about the relationship between a function and its first derivative, second derivative, or the combination of the two derivatives. Furthermore, the study examined students' thinking about open intervals where a function is positive or negative. Analysis of student work and verbal responses given by students while working on two tasks revealed that a majority of the 10 students who participated in the study were able to make sense of the relationship between a function and its first derivative. However, most of the students demonstrated a limited understanding of the relationship between a function and its second derivative. Additionally, most of the students exhibited a limited understanding of what it means for a function to be positive or negative on an open interval. Taken together, the students exhibited weak quantitative and covariational reasoning abilities. This is particularly concerning because calculus often serves as a gateway course for many STEM (Science, Technology, Engineering, and Mathematics) majors. Specifically, weak quantitative and covariational reasoning abilities such as those exhibited by the students in the current study could not only lead to high failure rates in calculus, but also to attrition from STEM majors. Implications for calculus instruction are included.

Keywords: Quantitative reasoning, covariational reasoning, functions, derivatives, calculus

Introduction

Oehrtman et al. (2008) argued that "the concept of function is central to undergraduate mathematics, foundational to modern mathematics, and essential in related areas of the sciences" (p. 27). On a related note, the concept of function is essential to the study of first-semester calculus (Calculus I) in the United States. Much research has reported on students' thinking about the concept of function over the last few decades (Beichner, 1994; Carlson, 1998; Dubinsky & Wilson, 2013; Habre & Abboud, 2006; Haciomeroglu et al., 2010; Malambo et al., 2018; Martínez-Plunell & Gaisman, 2012; Melhuish et al., 2020; Monk, 1992; Monk & Nemirovsky, 1994; Trujillo et al., 2023; Yerushalmy & Swidan, 2012). In fact, some of this research has examined students' thinking about the relationship between a function and its first derivative or the relationship between a function and its second derivative (Burns-Childers & Vidakovic, 2018; Carlson et al., 2002; Çetin, 2009; García-García & Dolores-Flores, 2021; Tsamir & Ovodenko, 2013). There is, however, a dearth of research that has investigated students' thinking about the relationship between a function's first derivative and the function's second derivative, as well as research that has examined students' understanding of open intervals where a function is positive or negative, which is the primary motivation for the present study. Understanding open intervals where a function is positive or negative is a fundamental skill, as it is foundational for understanding calculus ideas such as open intervals where a function is increasing or decreasing. Additionally, this study examined students' ability to sketch the graph of the second derivative when presented with the graph of a function, an area of research that has not been well explored.

Most of the existing literature on students' thinking about functions or their derivatives is situated in motion contexts (e.g., Altindis, 2025; Carlson et al., 2002; Kertil, 2020; Mkhatshwa, 2020; Nagle et al., 2017; Nguyen et al., 2025; Swidan et al., 2019). A growing number of researchers have called for the examination of students' thinking about derivatives in non-kinematics contexts (e.g., Biza et al., 2022; Jones, 2017; Kertil & Dede, 2022). In fact, Jones (2017) argued that "given the wide range of applications in science and engineering that are not based on kinematics … it is important to know how students understand applied derivatives in non-kinematics contexts." (p. 95). In an effort to contribute toward addressing the gap in knowledge identified by Jones (2017), the present study examined students' thinking about derivatives in two non-kinematics contexts, namely volume of water in a tank and the spread of a disease. This examination takes place through the lens of quantitative reasoning and covariational reasoning, two modes of reasoning that have long been identified by influential mathematics educators as essential for students hoping to understand quantities and relationships among quantities that can be represented using functions (Carlson et al., 2002; Thompson, 2011).

The primary goal of the study was to gain insight into students' quantitative reasoning and covariational reasoning in the context of working with functions situated in real-world, non-kinematics contexts. The secondary goal of the study was to gain insight into students' understanding of open interval where a function is positive, negative, or zero. Thus, taken together, the present study aims to answer the following research questions:

1. What does calculus students' thinking about the relationship between a function and its first and second derivative reveal about their quantitative reasoning?
2. What does calculus students' thinking about the relationship between a function and its first and second derivative reveal about their covariational reasoning?
3. What does calculus students' thinking about the relationship between a function and its first and second derivative reveal about their understanding of open intervals where a function is positive, negative, or zero?

Theoretical Underpinnings: Quantitative and Covariational Reasoning

This study uses the theoretical lenses of quantitative reasoning and covariational reasoning.

Quantitative reasoning

"Quantity" refers to a measurable attribute of an object (Thompson, 1993; 1994b). A water tank is one example of an object used in the present study, while the volume of the tank is an example of a quantity used in the same study. A key element of quantitative reasoning involves making sense of quantities or relationships between quantities (Smith & Thompson, 2007; Thompson, 1993). Making sense of quantities entails assigning numerical values to quantities, a process Thompson (1994d) refers to as quantification. Additionally, making sense of quantities entails reasoning about their units of measure. Furthermore, interpreting quantities is an important aspect of quantitative reasoning.

Covariational reasoning

"Covariational reasoning" refers to how two variables are changing in tandem in a dynamic situation or event. Carlson et al. (2002) described five levels of covariational reasoning (coordination, direction, quantitative coordination, average rate, and instantaneous rate) that are based on the mental actions reproduced in Table 1.

Table 1. Mental Actions of Covariational Reasoning (Reproduced from Carlson et al., 2002, p. 357)

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other [variable].	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x).
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable.	• Constructing an increasing straight line. • Verbalizing an awareness of the direction of change of the output while considering changes in the input.
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable.	• Plotting points/constructing secant lines. • Verbalizing an awareness of the amount of change of the output while considering changes in the input.
Mental Action 4 (MA4)	Coordinating the average-rate - of change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain. • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input.
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.	• Constructing a smooth curve with clear indications of concavity changes. • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct).

The aforementioned levels of covariational reasoning are increasingly sophisticated in the sense that a student engages in the lowest level of covariational reasoning (coordination) if they exhibit behaviors associated with the first mental action (MA1), while engaging at the second level of covariational reasoning (direction) requires exhibiting behaviors associated with the first two mental actions (MA1 and MA2), and so forth.

Related Literature

This section provides an overview of the of the existing literature that is related to the present study, namely students' understanding of functions, students' understanding of the first derivative, and students' understanding of the second derivative.

Students' understanding of functions

The importance of functions at the K-16 level has been underscored by several influential bodies of education and reputable mathematics education researchers (Dubinsky & Wilson, 2013; Monk, 1994; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000; Oehrtman et al., 2008; Thompson, 1994c). One reoccurring theme from the literature that has examined students' thinking about functions is that in as much as functions can be represented in multiple ways such as using graphs, numerical tables of values, or equations/formulas, students have a proclivity to work with functions in algebraic form (Habre & Abboud, 2006; Herman, 2007; Ibrahim & Rebello, 2012; Thompson, 1994b). On a related note, Thompson (1994c) posited that while "tables, graphs and expressions might be multiple representations of functions" (p. 23) from the perspective of mathematics instructors or researchers, they are not "multiple representations of anything to students" (p. 23). Another common finding from research that has investigated students' thinking about functions is that making sense of graphs of functions or creating graphs of functions is often a challenge for students (Beichner, 1994; Dominguez et al., 2017; Carlson et al., 2002; Monk, 1994). In a similar vein, other research indicates that students tend to envision graphs of functions as pictures of physical events, an idea that Monk (1992) refers to as iconic translation. Lastly, findings from research on students' reasoning about functions indicate that translating information from one representation of a function to another, such as selecting the correct kinematics graph that matches a given textual description about a functional situation, is sometimes problematic for students (Beichner, 1994; Martínez-Planell & Gaisman, 2012; Van den Eynde et al., 2019).

Students' understanding of the first derivative

A number of studies have reported on students' thinking about the first derivative. One major theme that emerges from this research is that students are prone to confusing a function's instantaneous rate of a change with either the function's output value or the accumulation of the function's output (Jones, 2017; Mkhatshwa, 2020, 2023; Mkhatshwa & Doerr, 2018; Prince et al., 2012; Rasmussen & Marrongelle, 2006). For instance, Mkhatshwa and Doerr (2018) reported on students who confused marginal cost (the rate of change of cost) with total cost (the accumulation of cost), while Rasmussen and Marongelle (2006) reported on students who did not make a conceptual distinction between the rate of change in the amount of salt and the amount of salt. On a related note, other findings indicate that students have a penchant for confusing units of derivative quantities such as speed with units of amount quantities such as distance (Lobato et al., 2012; Mkhatshwa & Doerr, 2018; Mkhatshwa, 2020). Other research has reported on students' difficulties related to sketching the graph of the first-derivative function when given the graph of a function (García-García & Dolores-Flores, 2021; Orhun, 2012; Ubuz, 2007).

Students' understanding of the second derivative

Research on students' thinking about the second derivative is limited. A common finding of the few available studies is that making sense of inflection points is not only challenging for calculus students, but also for students who have completed a course in ordinary differential equations (Carlson et al., 2002; Jones, 2019; Ovodenko & Kouropatov, 2019; Tsamir & Ovodenko, 2013). Carlson et al. (2002) reported on students who had "difficulty forming images of continuously changing rate and [students who] could not accurately represent or interpret inflection points or increasing and decreasing rate for dynamic situations" (p. 352). Some of the students in Tsamir and Ovodenko's (2013) study confused inflection points with local extrema, an observation also made by Jones (2019). Additionally, none of the 52 students who participated in Tsamir and Ovodenko's (2013) study "correctly identified all inflection points on the five graphs" (p. 416) of the functions they were presented with. Jones (2019) reported that some of the students who participated in his study "struggled to use appropriate aspects of covariational reasoning to make sense of concavity and inflection points" (p. 541) in various real-world contexts, including economics. On a positive note, Ovodenko and Kouropatov (2019) proposed a teaching unit that has the potential to address common student misconceptions or errors as it relates to the concept of an inflection point. The unit utilizes several technological and educational tools, including GeoGebra, a dynamic mathematics software.

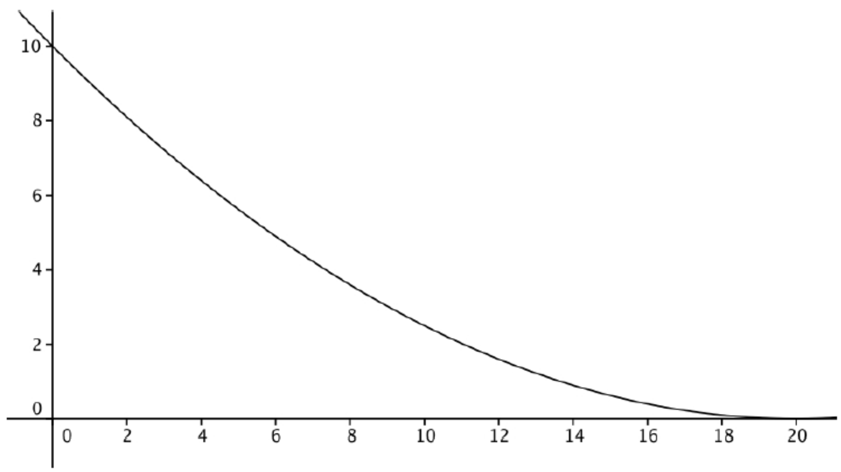
In summary, the aforementioned bodies of literature have established that the concept of function is generally not well-understood by students. Specific student challenges include making sense of functions in multiple representations and a proclivity for working with functions in algebraic form. There is, however, a dearth of research that has explored students' thinking about functions in connection to open intervals where functions are positive, negative, or zero, a knowledge gap that the present study seeks to address in the literature. The concept of open intervals where a function is positive, negative, or zero is worthy of greater attention because it is foundational for understanding fundamental concepts or topics in the study of first-semester calculus, including optimization problems, curve sketching, critical points of a function, the first derivative test, and the intermediate value theorem. Furthermore, the literature has established that students have difficulties making sense of relationships between functions and their derivatives, especially in motion contexts. The present study examines students' thinking about functions and their derivatives in real-world and non-motion contexts to understand how students' reasoning in these contexts might inform the field of calculus education. Finally, there is a paucity of research that has examined how students make sense of the relationship between a function's first derivative and a function's second derivative, and how students sketch graphs of second derivative functions when given the graph of a function. The present study uses the lens of quantitative reasoning and covariational reasoning to address the previously noted knowledge gaps in the literature.

Methodology

Research design

This exploratory study utilized a qualitative research design (Creswell, 2014). Specifically, to answer the research questions listed in the Introduction section, I used structured task-based interviews (Goldin, 2000). According to Goldin, "the design of structured task-based interviews needs to take into account their research purposes" (p. 519). He added, the research purposes "may include (for example) exploratory investigation; refinement of observation, description, inference, or analysis techniques; development of constructs and conjectures; investigation or testing of advance hypotheses; and/or inquiry into the applicability of a model of teaching, learning, or problem solving." (p. 519). In the present study, I used structured task-based interviews with the sole purpose of exploring students' quantitative or covariational reasoning in the context of working with functions and their derivatives in real-world non-motion contexts. In designing or conducting the task-based interviews, I followed key principles and techniques outlined by Goldin (2000), including choosing "tasks that are accessible to the participants" (p. 540), designing "task-based interviews to address advance research question" (p.540), and encouraging "free problem solving" (p. 542). The interviews covered two tasks:

Task 1 (Volume of water in a tank): The graph shows the volume $f(t)$ (in gallons) of water in a tank t seconds after a drain hole is unplugged.



a) What parts of the domain of f correspond to f being positive, negative, or zero? Explain.

b) Based on your observations in part a), how can you interpret f in the context of volume of water in the tank and time? Explain.

c) What parts of the domain of f correspond to f' being positive, negative, or zero? Explain.

d) Based on your observations in part c), what does f' tell you about f in the context of volume of water in the tank and time? Explain.

e) What parts of the domain of f correspond to f'' being positive, negative, or zero? Explain.

f) Based on your observations in part e), what does f'' tell you about f in the context of volume of water in the tank and time?

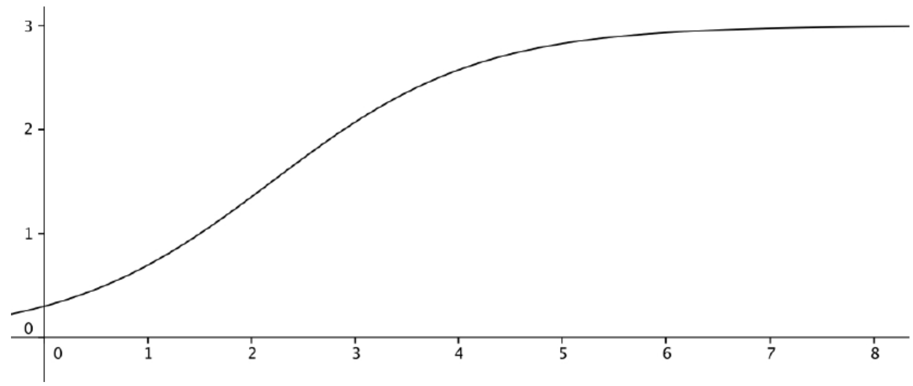
g) Considering the domain of f, what does f'' tell you about f' in the context of volume of water in the tank and time?

h) What are the units of the quantities represented by f' and f''?

i) Construct a rough sketch of the graph of f'.

j) Construct a rough sketch of the graph of f''.

Task 2 (Spread of a disease): The graph shows the number $f(t)$ (in hundreds) of laboratory mice that have become sick with the flu by the t^{th} day after the start of an experiment.



a) What parts of the domain of f correspond to f being positive, negative, or zero? Explain.

b) Based on your observations in part a), how can you interpret f in the context of the spread of the disease and time? Explain.

c) What parts of the domain of f correspond to f' being positive, negative, or zero? Explain.

d) Based on your observations in part c), what does f' tell you about f in the context of the spread of the disease and time? Explain.

e) What parts of the domain of f correspond to f'' being positive, negative, or zero? Explain.

f) Based on your observations in part e), what does f'' tell you about f in the context of the spread of the disease and time?

g) Considering the domain of f, what does f'' tell you about f' in the context of the spread of the disease and time?

h) What are the units of the quantities represented by f' and f''?

i) Construct a rough sketch of the graph of f'.

j) Construct a rough sketch of the graph of f''.

Each participant completed Task 1 prior to working on Task 2. At the conclusion of each task, the interviewer asked the following questions, in addition to clarifying questions (For example, what do you mean when you say f' is decreasing with a negative slope?) that were asked by the interviewer as students worked through Tasks 1 and 2: What was the easiest part when solving this problem? Explain. What was the difficult part when solving this problem? Explain.

The purpose of each task was four-fold. First, to examine students' reasoning about the relationship between a function (f) and its first derivative (f'). Second, to examine students' reasoning about the relationship between a function (f) and its second derivative (f''). Third, to examine students' reasoning about the relationship between the first derivative of a function (f') and the second derivative of a function (f''). Fourth, to examine students' reasoning about open intervals where a function is positive or negative. Average completion task time for each interview was 36 minutes (19 minutes on Task 1 and 17 minutes on Task 2). The shortest interview lasted for eight minutes while the longest interview lasted for 34 minutes.

Setting and participants

This study took place at a public research university in the United States. The subjects were 10 students (Pseudonyms Dean, Mary, Claire, Camila, Cooper, Nate, Kiara, Ken, Adrian, and Ada) who were enrolled in three sections of a traditional Calculus I course during the fall semester of 2022. The three sections were taught by two seasoned professors. It is worth noting that the researcher is not one of the professors. The reader may find additional information about the course and study participants from an earlier manuscript (Mkhatshwa, 2025a).

Data collection and analysis

There are two sources of data for the current study, namely transcriptions of interviews and work written by students during each interview session. The data were analyzed using a thematic analysis approach (Braun & Clark, 2006). According to Braun and Clark, thematic analysis is a method "for identifying, analysing and reporting patterns (themes) within data" (p. 79). In the context of the present study, a theme is a similar response to the prompts included in Tasks 1 and 2 that was given by at least two students. Specifically, the data were analyzed in three stages. In the first stage of the analysis, I examined students' quantitative reasoning. Specifically, I examined how students interpreted the quantities represented by the functions f, f', and f'' in the contexts of the two tasks. Furthermore, I examined how students reasoned about the units of the quantities represented by the functions f, f', and f''. Additionally, I examined how students reasoned about the relationship between the quantities represented by the functions f' and f''. In the second stage of the analysis, I examined students' covariational reasoning. In particular, I examined how students engaged (or did not engage) in the five levels of covariational reasoning described in the theoretical underpinnings section, namely coordination, direction, quantitative coordination, average rate, and instantaneous rate. In the third stage of the analysis, I examined how students reasoned about open intervals where the functions f, f', and f'' are positive, negative, or zero (i.e., neither positive nor negative). Table 2 shows examples of how the codes were applied.

Table 2

Sample codes from the data

Code	Description of Code	Examples
Interpreting (an aspect of quantitative reasoning) f' or f'' as a rate of change.	Using the word "per" when prompted to assign units of measure to the quantities represented by the functions f' or f'' in Task 1 or Task 2.	Researcher: What are the units of the quantities represented by f' [Task 1] and f'' [Task 1]? Dean: The units of f' is gallons per second and the units of f'' is gallons per second per second.
Attention to direction of change (the second level of covariational reasoning).	Describing how two quantities are changing simultaneously in Task 1 or Task 2.	Researcher: Based on your observations in part a) [Task 2], how can you intepret f in the context of the spread of the disease and time? Explain. Ken: As time goes on, more mice get sick.
Reasoning about positive functions.	Describing intervals where at least one of the functions f, f', or f'' in Task 1 or Task 2 is positive.	Researcher: What parts of the domain of f [Task 1] correspond to f'' being positive, negative, or zero? Explain. Mary: Between $t = 0$ and $t = 20$, the second derivative [f''] will be positive because the graph of $f'(t)$ is increasing as the slope of $f(t)$ decreases.

To ensure rigor and trustworthiness of the data analysis, I used peer debriefing (Spall, 1998). According to Spall, "peer debriefing supports the credibility of the data in qualitative research and provides a means toward the establishment of the overall trustworthiness of the findings" (p. 280). In the context of the present study, peer debriefing entailed discussing the research design, the theoretical underpinnings for the study, the tasks, and the emergent themes (findings) from the study with one mathematics education researcher. Discrepancies in emergent themes were resolved as part of the peer debriefing process. Permission (Project Reference Number: 04117e) to conduct this study was granted by Miami University's Research Ethics & Integrity Program.

Results

Students' quantitative reasoning

Overall, students' interpretations of the quantities represented by the functions f, f', and f'' across the two tasks suggests that while students are able to correctly interpret quantities represented by first derivatives in real-world non-kinematics contexts, interpreting quantities represented by second derivatives in the same contexts is particularly challenging for students. Additionally, analysis of students' work across the two tasks suggests that interpreting the relationship between the quantities represented by f' and f'' in each of the two tasks was problematic for a majority of the students who participated in this study. Table 3 summarizes students' interpretations of the quantities represented by the functions f, f', and f'' in the two tasks. The symbol ✓ indicates that a student correctly interpreted the indicated quantity (or relationship between quantities) and the symbol X means that a student incorrectly interpreted the indicated quantity (or relationship between quantities). Additionally, the symbol IDK indicates that a student stated that they "did not know" how to interpret the indicated quantity (or relationship between quantities). Furthermore, the symbol T1 indicates that the interpretation of the indicated quantity (or relationship between quantities) pertains to Task 1 whereas the symbol T2 indicates that the interpretation of the indicated quantity (or relationship between quantities) pertains to Task 2.

Table 3

A Summary of Students' Interpretations of Quantities Represented by the Functions f, f', and f''

Student	f		f'		f''		Relationship between f' and f''	
	T1	T2	T1	T2	T1	T2	T1	T2
Dean	✓	✓	✓	✓	✓	X	✓	✓
Mary	✓	✓	X	X	X	X	X	X
Claire	✓	X	✓	X	X	X	X	X
Camila	✓	✓	IDK	X	IDK	X	IDK	IDK
Cooper	✓	✓	✓	X	X	✓	X	X
Nate	X	✓	X	X	X	X	X	X
Kiara	✓	X	✓	X	X	X	X	X
Ken	✓	✓	✓	✓	X	X	X	✓
Adrian	✓	X	✓	✓	X	X	X	X
Ada	✓	X	✓	✓	IDK	X	IDK	X

As can be seen in Table 3, nine of the study participants correctly interpreted the quantity represented by the function f in Task 1 compared to six students who correctly interpreted the quantity represented by the function f in Task 2. To some extent, it can be argued that students' correct interpretations of the function f in both tasks is due to the fact that an explanation of the function f was provided as part of the description of each task.

Six students correctly interpreted the quantity represented by the function f' in Task 1 as the rate at which the volume of water is decreasing in the tank after the drain hole is unplugged, compared to four students who correctly interpreted the quantity represented by the function f' in Task 2 as the rate at which the number of sick mice is increasing over time. As an example of one of the incorrect interpretations given by students when prompted to explain the meaning of f' in the context of Task 1 (i.e., water flowing out through a drain hole), Nate remarked that "a first derivative [referring to the function f' in Task 1] would tell you the critical values where a function is increasing or decreasing, and the local maximum and minimum values of the graph [that appears in Task 1]." As an example of one of the incorrect interpretations given by students when prompted to explain the meaning of f' in the context of Task 2 (i.e., the spread of a disease), Kiara commented that the first derivative tells us that "as time increases, the disease also increases until it remains somewhat constant." It would seem that in her interpretation of the function f' in Task 2, Kiara paid attention to the physical features of the graph given in the task.

Two students correctly interpreted the quantities represented by the function f'' in both tasks. In particular, one student (Dean) correctly interpreted the quantity represented by the function f'' in Task 1 as how quickly the rate of water loss in the tank is changing over time and another student (Cooper) correctly interpreted the quantity represented by the function f'' in Task 2 as how quickly the rate of sick mice is changing over time. As an example of one of the incorrect interpretations given by students when prompted to explain the meaning of f'' in the context of Task 1, Adrian explained that the function f'' tells us that "the volume of water in the tank decreases as time increases." Like Kiara, it would seem that Adrian paid attention to the physical features of the graph presented in Task 2 when interpreting the quantity represented by the function f'' in the task. As an example of one of the incorrect interpretations given by students when prompted to explain the meaning of f'' in the context of Task 2, Claire noted that the function f'' tells us that "as time goes on, more mice will get sick."

Two students correctly interpreted the relationship between the quantities represented by the functions f' and f'' in at least one of the two tasks. One of these students (Dean) correctly interpreted the relationship between the quantities represented by the functions f' and f'' in both tasks. When prompted to explain what f'' tells us about f' in the context of Task 1, Dean explained that "the speed of the decrease of water is slowing down, which means that the volume of water in the tank decreases slower and slower in a certain time period." Similarly, when prompted to explain what f'' tells us about f' in the context of Task 2, he reasoned that "the speed of the spread of the disease increases at the beginning and decreases from the second day [of the outbreak]." Most of the students who incorrectly interpreted the relationship between the quantities represented by the functions f'' and f' either remarked about the physical features of the graphs shown in each task, or they provided responses that I coded as not making sense. One such response that I coded as not making sense was given by Nate who reasoned that "the f'' in context with f' would give you a better representation of the relationship between volume of water in the task [Task 1] and time." In response to the same prompt regarding Task 2, Nate's response suggest that he paid attention to the physical features of the graph given in the task in an effort to make sense of the relationship between the quantities represented by the functions f'' and f'. Specifically, he explained that "the f'' tells you about f' that as the number of days increase, the number of sick mice increases reaching a peak on the eighth day."

Only four students correctly determined the units of measure for the quantity represented by the function f' in Task 1 as "gallons per second", compared to only two students who correctly determined the units of measure for the quantity represented by the function f' in Task 2 as "sick mice per day." Common incorrect units of measure assigned by students to the quantity represented by the function f' in Task 1 include "gallons" or "gallons per second squared", whereas common incorrect units of measure assigned by students to the quantity represented by the function f' in Task 2 include "number of sick mice" or simply "number of mice." Likewise, four students correctly determined the units of measure for the quantity represented by the function f'' in Task 1 as "gallons per second squared", compared to two students who correctly determined the units of measure for the quantity represented by the function f'' in Task 2 as "sick mice per day squared." Typical incorrect units of measure assigned by students to the quantity represented by the function f'' in Task 1 include "gallons" or "time in seconds" whereas typical incorrect units of measure assigned by students to the quantity represented by the function f'' in Task 2 include "hundreds of mice per day" or "number of mice."

Students' covariational reasoning

In assessing students' covariational reasoning, I focused on how students constructed first-derivative graphs and second-derivative graphs corresponding to the graphs of the functions that are presented in each of the two tasks. Because students' covariational reasoning is similar in each task, regardless of which derivative graph (first or second) they constructed, I limit my reporting in this section to how students engaged (or did not engage) in covariational reasoning while creating first-derivative graphs corresponding to the functions given in the two tasks.

An examination of students' covariational reasoning in the two tasks revealed that five students engaged in the highest level of covariational reasoning (i.e., instantaneous rate) while working on Task 1, compared to only three students who engaged in the same level of covariational reasoning while working on Task 2. Ada's graph (shown in Figure 1) is representative of students' graphs that I coded as showing evidence of engaging in the highest level of covariational reasoning in Task 1, while Dean's graph (shown in Figure 2) is representative of students' graphs that I coded as showing evidence of engaging in the highest level of covariational reasoning in Task 2.

Figure 1
Ada's First-Derivative Graph in Task 1

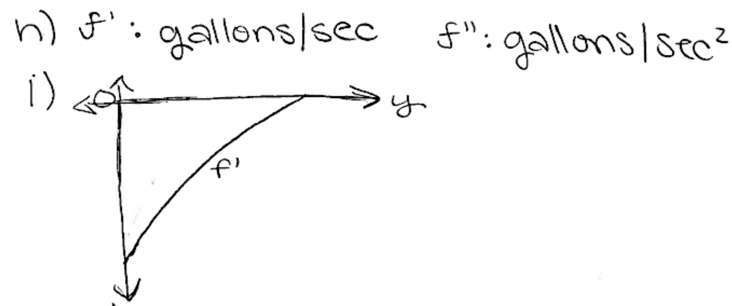
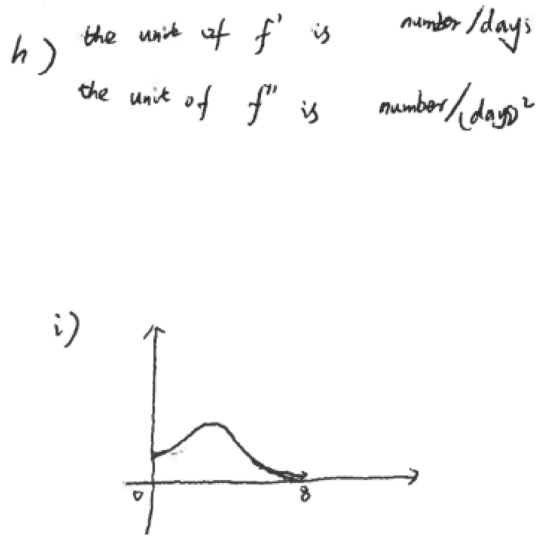


Figure 2
Dean's First-Derivative Graph in Task 2



Ada and Dean created vertical and horizontal axes for their graphs, which is evidence of engaging in the first level of covariational reasoning (i.e., coordination). I note that even though Dean's axes are not labeled, the student verbally and correctly stated that the quantity [time] on the horizontal axis would be measured in seconds while the quantity [hundreds of sick mice per day] on the vertical axes would be measured in number [of sick mice] per day. Ada correctly stated that the units of the quantity [time] on the horizontal axis would be measured in seconds, while that of the quantity [flow rate] on the vertical axis would be measured in gallons per second. Both students expressed verbal awareness of the quantities represented in the vertical axes of their graphs as changing (i.e., decreasing in the case of Task 1 and increasing in the case of Task 2) with time, thus exhibiting evidence of engaging in the second level of covariational reasoning (i.e., direction). For example, while commenting on the domain of the graph of the function given in Task 1, Ada remarked that "as time progresses, the volume of water in the tank decreases." The students, especially Dean, plotted a few points that they initially joined with contiguous line segments prior to using smooth curves to join the points, thus providing evidence of engaging in levels 3, 4, and 5 of covariational reasoning. Specifically, the plotting of points is evidence of engaging in the third level of covariational reasoning also known as the quantitative coordination level. Furthermore, the fact that the students initially joined the points they plotted using "contiguous secant lines" is evidence of engaging in the fourth level of covariational reasoning (also known as the average rate level). Additionally, the students engaged in the highest level of covariational reasoning (i.e., the fifth level of covariational reasoning, also known as the instantaneous rate level). For instance, when asked to explain the meaning of the function f' in Task 2, Dean remarked that "the number of infected mice is increasing rapidly at the very beginning days and then the number of infected mice increases slower." He added, "there are fewer mice being infected after the eighth day of the experiment." I argue that Dean's attention to how the number of mice infected grows rapidly initially and then slows down after the eighth day of the experiment demonstrates the student's awareness of the instantaneous rate of change of the function considered in the task. Moreover, the student's graph [Figure 2] is a smooth curve with clear indications of concavity changes, which is yet another evidence of engaging in the highest level of covariational reasoning.

The rest of the students, at best, engaged at the second level of covariational reasoning while working on the two tasks. Nate's graph (shown in Figure 3) is representative of students' graphs that I coded as showing evidence of engaging in the first or second level of covariational reasoning in Task 1, while Cooper's graph (shown in Figure 4) is representative of students' graphs that I coded as showing evidence of engaging in the first or second level of covariational reasoning in Task 2.

Figure 3

Nate's First-Derivative Graph in Task 1

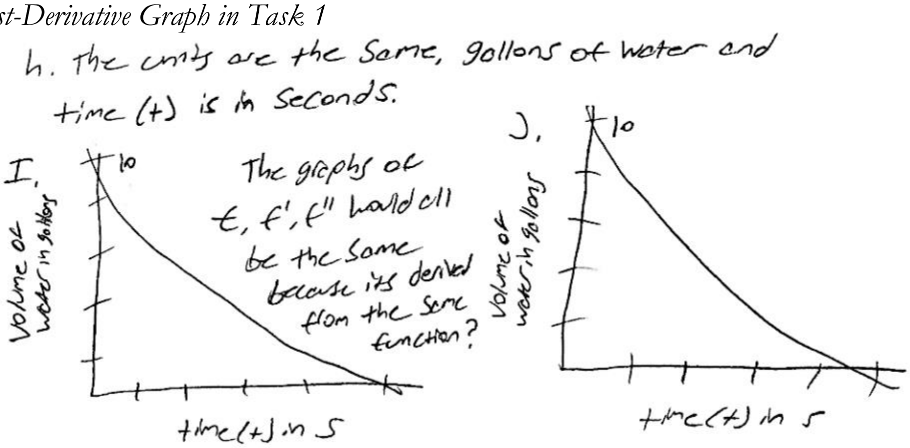
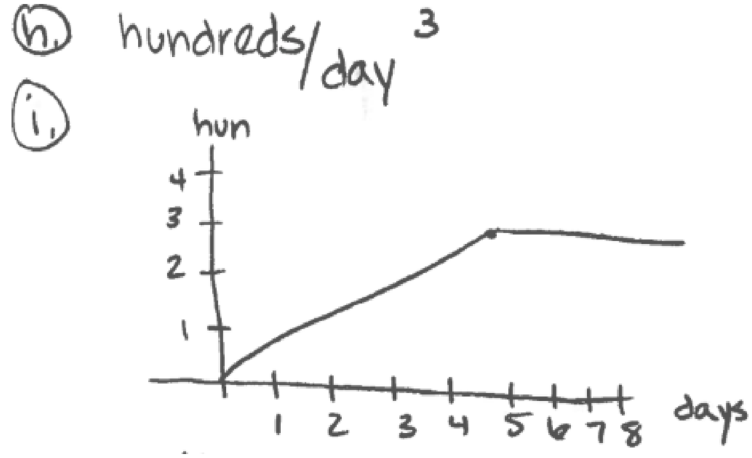


Figure 4
Cooper's First-Derivative Graph in Task 2



The fact that both Nate and Cooper created and labeled horizontal and vertical axis for their graphs demonstrates an awareness that two quantities are changing in tandem in each task (e.g., the volume of water and time in Task 1). This is evidence of engaging in the first level of covariational reasoning. Since Nate's first-derivative graph (i.e., the graph on the left and labeled with the variable I in Figure 3) appears to be the same as the graph that is presented in the task (i.e., the original function), I judged his graph as not exceeding the first level of covariational reasoning. On a related note, I note that Nate's first-derivative graph has the same units of measure as that of the original graph, further suggesting that his first-derivative graph is simply a reproduction of the original graph. While not relevant, I note that the graph on the left in Figure 3 is Nate's second-derivative graph which also appears to be a reproduction of the original graph. When prompted to talk about the challenges he encountered while working on Tasks 1 and 2, Nate repeatedly remarked on the difficulty of sketching the graphs of f' and f'' in the aforementioned tasks:

Researcher: What was the difficult part when solving this problem [Task 1]? Explain.

Nate: The difficult part was not having an equation for $f(x)$, and so it's hard to state what the context or relationship between f, f', and f'' would be and what the graphs of $f'(x)$ and $f''(x)$ would look like.

Researcher: What was the difficult part when solving this problem [Task 2]? Explain.

Nate: The difficult part was that there was no equation for $f(x)$, so it was difficult to determine the sketches of the graphs for $f'(x)$ and $f''(x)$, so I assumed they [graphs of $f'(x)$ and $f''(x)$] were the same [as the graph of $f(x)$].

A close examination of Nate's difficulties as it relates to sketching the graphs of f' and f'' in each task reveals that his difficulties were two-fold. First, Nate repeatedly remarked on not having an equation for the functions f. Second, Nate was not able to make sense of how the functions f, f', and f'' were all related in the context of each of the tasks, especially in Task 1. This is indicated by his claim, "so it's hard to state what the context or relationship between f, f' and f'' would be" when he commented on how the graphs of f' and f'' would look like in Task 1.

When asked to sketch the graph of the second derivative in each task, a majority of the students either reproduced the graph they sketched for the first derivative, reproduced the graph of the original function given in each task, or they stated that they "did not know how to sketch the graph of the second derivative." One of the students, Claire, claimed that she struggled with envisioning how the graph of f'' would look like. To substantiate her claim, she reasoned, "we have not spent enough time in class graphing them [graphs of f'']." I considered Cooper's use of two connected line segments in Figure 4 (one increasing and another constant) as evidence of engaging in the second level of covariational reasoning. Specifically, the increasing line segment demonstrates that the student recognized that at the beginning of the outbreak (within the first four and a half days of the outbreak per the evidence contained in Figure 4), the number of sick mice would increase. This is evidence of engaging in the direction of change level (i.e., the second level of covariational reasoning).

Students' reasoning about open intervals where a function is positive, negative, or zero

I begin this section by noting that the domain of each of the functions (f, f', and f''') in Task 1 is considered to be the open interval $(0,20)$, while the domain of each of the functions (f, f', and f'') in Task 2 is considered to be the open interval (0,8). I further note that with regard to Task 1, the functions f and f'' are considered to be positive on the open interval $(0,20)$, while the function f' is considered to be negative on the open interval $(0,20)$. With regard to Task 2, the functions f and f' are considered to be positive on the open interval $(0,8)$, while the function f'' is considered to be positive on the open interval $(0,2.5)$ and negative on the open interval $(2.5,8)$. Furthermore, the function f'' in Task 2 is the only function that is considered to be zero (neither positive nor negative) in some part of its domain. In particular, this happens when the input variable t takes on the value of 2.5 (i.e., when $t = 2.5$). I remark that only four students correctly determined the input value of $t = 2.5$ as a location in the graph of the function f'' where the same function would be considered to be zero.

Analysis of students' understanding of intervals over which the functions f, f', and f'' in each task are positive, negative, or zero revealed that at least half of the students correctly determined the open intervals over which the functions f are positive or negative in each task, and that a majority of the students were not able to correctly identity open intervals over which the functions f' and f'' are positive or negative in each task. Specifically, seven students correctly determined the open interval over which the function f is positive or negative in Task 1, compared to five students who correctly determined the open interval over which the function f is positive or negative in Task 2. Additionally, six students correctly determined the open interval over which the function f' is positive or negative in Task 1, compared to only four students who correctly determined the open interval over which the function f' is positive or negative in Task 2. Furthermore, only 4 students correctly determined the open interval over which the function f'' is positive or negative in Task 1, compared to only three students who correctly determined the open intervals over which the function f'' is positive or negative in Task 2. The following excerpt highlights some of the challenges that students cited as stumbling blocks in their attempt to determine open interval where the functions f, f', and f'' in each task are positive or negative.

Researcher: What was the easiest part when solving this problem [Task 1]? Explain.

Kiara: Determining the relationship between f, f', and f'' to the volume of water and time because I was able to look at the graph of what is happening [Referring to the graph of the function f that is given as part of the task].

Researcher: What was the difficult part when solving this problem [Task 1]? Explain.

Kiara: Determining the parts of the domain corresponding to f [the function given on the task] being negative, positive, or zero because there was no equation given.

Researcher: What was the easiest part when solving this problem [Task 2]? Explain.

Kiara: Relating the number of sick mice to the number of days past because it was represented by the graph [i.e., understanding how the number of sick mice changed over time based on the information given on the graph].

Researcher: What was the difficult part when solving this problem [Task 2]? Explain.

Kiara: Drawing the graphs for f' and f'' because no exact numbers were given to help with the graph [the graph given in the task].

Based on the evidence presented in the preceding excerpt, it would seem that making sense of the relationship between the quantities represented by the input and output variables for the function f was straightforward for Kiara in each task. Kiara identified "determining the parts of the domain corresponding to f being negative, positive, or zero" as being challenging in Task 1. That is, determining open intervals where the function f is positive or negative was particularly challenging for Kiara. She went on to identify sketching the graphs of the functions f' and f'' as problematic in Task 2, an experience shared by several other students as well.

When prompted to comment on difficulties they had with determining open intervals where the functions f, f', and f'' are positive or negative in each task, at least five students remarked on: (1) the challenge of not being given an algebraic formula/equation for the functions f' and f'' in each task, (2) the challenge of not being given a graph for the functions f' and f'' in each task, (3) the challenge of not knowing what it means for a function to be positive or negative in its domain, or (4) they cited the lack of labels on the vertical and horizontal axes of the graphs given in Tasks 1 and 2 as something that limited them from being able to correctly determine the open intervals where the functions f, f', and f'' in each task are positive or negative. It should be noted that even though the vertical and horizontal axes for the graphs given in Tasks 1 and 2 are not labeled, the information needed to label the aforementioned axes is provided in the opening statement of each task. In fact, one student (Claire) explicitly stated that as a consequence of the aforementioned challenges, she relied on "guessing" in her effort to determine open intervals where the functions f, f', and f'' in each task are positive or negative.

Discussion and Implications

This section presents a discussion of the key findings from the current study. Specifically, these findings are discussed considering the literature reviewed earlier. First, while several students correctly interpreted the quantity represented by the function f' in each task, a majority of the study participants not only found interpreting the quantity represented by the function f'' in each task to be problematic, but they also incorrectly interpreted this quantity. In essence, this finding suggests that while interpreting real-world non-kinematics quantities that can be represented using first derivatives is seemingly straightforward for students, the same cannot be said with regard to interpreting similar quantities that can be represented using second derivatives. A few other studies have reported on calculus students' difficulties related to interpreting the second derivative (Carlson et al., 2002; Jones, 2019; Ovodenko & Kouropatov, 2019; Tsamir & Ovodenko, 2013). Furthermore, students' reasoning about units of measure for quantities represented by the functions f, f', and f'' in each task suggests that a majority of the students confused amount quantities with other amount quantities, or they confused amount quantities with rate quantities. Similar results have been reported by other researchers (Jones, 2017; Mkhatshwa, 2020, 2023; Mkhatshwa & Doerr, 2018; Prince et al., 2012; Rasmussen & Marrongelle, 2006). Taken together, the study participants exhibited limited quantitative reasoning abilities as evidenced by the challenges they reported when tasked with interpreting real-world non-kinematics quantities represented by second derivatives or making sense of units of measure for these quantities. From an instructional point of view, I recommend that calculus instructors be intentional about incorporating classroom activities or assignments (the teaching unit on making sense of inflection points that was proposed by Ovodenko and Kouropatov, 2019) that can help students develop strong quantitative reasoning skills, especially as it relates to interpreting quantities represented by first or second derivatives, in addition to analyzing units of measure for these quantities. This is especially important because calculus instruction in the United States tends to focus more on helping students develop strong calculational/algebraic knowledge (e.g., calculating derivatives) and less on important aspects of quantitative reasoning such as interpreting quantities represented by derivatives or the units of measure for such quantities.

Second, a majority of the students in the present study exhibited limited covariational reasoning skills. Specifically, only three students engaged at the highest level of covariational reasoning when tasked with creating first-derivative graphs in the two tasks. Other research has reported on students' difficulties with engaging in covariational reasoning when solving calculus tasks (Jones, 2017, 2019; Mkhatshwa, 2023). Recent findings by Engelke-Infante (2021) suggests that the creation and use of diagrams during calculus instruction could help students develop strong covariational reasoning skills. Considering that covariational reasoning is an important mode of reasoning that is essential for students hoping to acquire a solid understanding of calculus ideas such as inflection points (Carlson et al., 2002), I urge that calculus instructors and calculus textbook authors should strive to include, in their classes or textbooks as the case may be, assignments that are geared toward supporting students in enhancing their covariational reasoning abilities. Recent studies have reported that opportunities to engage in covariational reasoning that are included in widely used calculus textbooks in the United States are not only minimal but are often limited to the lowest levels of covariational reasoning (Mkhatshwa, 2022, 2025b). Increasing covariational reasoning opportunities in calculus textbooks is paramount as research indicates that the content that instructors teach in the classroom and how they teach it is often determined by course textbooks (Begle, 1973; Reys et al., 2004; Wijaya et al., 2015).

Third, a majority of the students struggled with determining open intervals where the functions f, f', and $f''$are positive or negative in each of the two tasks they worked on. Among other difficulties, students reported that not having access to the algebraic or graphical forms of the aforementioned functions as a source of their difficulty in determining the open intervals where these functions are positive or negative. Previous studies have reported on students' proclivity to working with functions in algebraic form (Habre & Abboud, 2006; Herman, 2007; Ibrahim & Rebello, 2012; Thompson, 1994b). Additionally, other students in the present study cited not knowing what it means for a function to be positive or negative as the main reason they failed to determine open intervals where the functions f, f', and f'' are positive or negative. Taking into consideration that the study participants were calculus students and that finding open intervals where a function is positive or negative is content typically covered in courses (e.g., college algebra or precalculus) that are considered prerequisite to the study of calculus, findings of the present study suggest that students' struggles with making sense of open intervals where a function is positive or negative persists through at least the study of calculus. Thus, it might be important for calculus instructors not to assume that this content is well understood by students prior to the study of calculus. In essence, I recommend that calculus instructors include this concept among other concepts (e.g., the concept of function) they typically review with students prior to teaching concepts that are specifically about the study of calculus.

Major Contributions of this Study to the Literature

In summary, and as a major contribution to the literature, findings of the present study suggest that students' difficulties with quantitative reasoning or covariational reasoning are not only limited to kinematics contexts (e.g., Carlson et al., 2002, Mkhatshwa, 2020), but that these difficulties also extend to non-kinematics contexts such as the contexts of volume of water in a tank and the spread of a disease, respectively, used in the present study. In particular, findings of this study indicate that interpreting quantities represented by the second derivative (f'') in non-kinematics contexts, an important aspect of quantitative reasoning, is particularly challenging for first-semester calculus students. Furthermore, findings of this study show that engaging in higher levels of covariational reasoning is not only particularly challenging for students when working with kinematics contexts (e.g., Carlson et al., 2002), but also when they are working with non-kinematics contexts.

In terms of moving the field forward, the observed widespread students' difficulties as it relates to making sense of derivatives in motion and non-motion contexts generally points to weaknesses in students' conceptual understanding of the concept of the derivative. Specifically, these weaknesses point to a need to focus on effective instructional approaches (Mkhatshwa, 2024) in the teaching of the concept of the derivative. As detailed in Mkhatshwa (2024), these effective instructional approaches often involve integrating educational technologies (e.g., Desmos and GeoGebra) and problem-solving strategies in the teaching of the concept of the derivative. From a research standpoint, the students' weaknesses observed in the present study point to a need for calculus education researchers to conduct design-based research (DBR). In particular, the goal of the DBR would be to develop teaching interventions (and assessing the effectiveness, or lack thereof, of these teaching interventions) that could be adopted by calculus instructors to help students overcome the weaknesses observed in the present study.

Considering that there is a dearth of research that that has examined students' thinking about open intervals where a function is negative, zero, or positive, findings of this study as it relates to the aforementioned line of research provide valuable insight for both researchers and calculus instructors. Specifically, findings of this study provide a baseline for other researchers who may be interested in further research as it relates to students' thinking about open intervals where a function is negative, zero, or positive. For calculus instructors, findings of this study provide an insight on the nature of students' difficulties as it relates to open intervals where a function is negative, zero, or positive. Consequently, calculus instructors may want to consider helping students develop a solid understanding of these precalculus ideas, prior to covering calculus-specific topics such as those dealing with finding open intervals where a function is increasing or decreasing.

Limitations

I remark on two limitations of the current study. First, the study participants were recruited from the same institution, something that limits the study's generalizability (i.e., the study's external validity). Taking into consideration that they were recruited from three sections of a Calculus I course taught by two experienced professors, I argue that the severity of this limitation is somewhat alleviated. Second, the sample size is relatively small, another potential threat to the study's external validity. I note that in conducting this exploratory study, the goal was not so much to extend the results to other settings, but rather to gain insight on calculus students' thinking about the concept of function through the lens of quantitative and covariational reasoning. That is, the goal of the study what to determine which aspects of the concept of the derivative are more or less challenging to calculus students. Future research may utilize a larger sample size and study participants from several institutions to improve the present study's external validity.

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Thembinkosi Peter Mkhatshwa (mkhatstp@miamioh.edu) is an Associate Professor of Mathematics at Miami University-Middleton, Ohio, United States. He teaches mathematics and statistics courses. His research generally spans three areas of study, namely student' thinking about fundamental ideas/concepts in calculus, experts' views on effective instructional approaches in the teaching of fundamental ideas/concepts in calculus, and opportunities to learn about fundamental ideas/concepts provided by widely used calculus textbooks in the United States.

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