

Prospective Mathematics Teachers' Perceptions of Infinity and Their Academic Knowledge About the Arithmetic of Infinity

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ABSTRACT

This study examines prospective mathematics teachers' perceptions of infinity and their academic knowledge of the arithmetic of infinity. A concurrent nested mixed-methods design was employed. The participants consisted of 100 volunteer third- and fourth-year prospective teachers enrolled in a mathematics teacher education program at a state university in the Aegean Region of Türkiye during the 2024–2025 academic year. Data were collected in a classroom setting via an online form comprising one open-ended question and 24 multiple-choice items. The open-ended question aimed to elicit participants' conceptual perceptions of infinity, while the multiple-choice items assessed their academic knowledge of arithmetic operations involving infinity. In particular, participants' understandings of operations involving ∞ with ∞ , 1, 0, and π were examined separately. Qualitative data were analyzed using descriptive analysis, whereas quantitative data were analyzed using descriptive statistics. The findings indicate that prospective teachers' perceptions of infinity are predominantly grounded in potential infinity rather than actual infinity. Although their overall performance on arithmetic operations involving infinity was above average, notable difficulties emerged, particularly in operations involving ∞ and 0. The results further reveal that intuition-based perceptions of infinity negatively reflect prospective teachers' procedural performance in operations involving infinity. When interpreted through the combined lenses of conceptual change theory and APOS theory, the findings suggest that prospective teachers have not successfully undergone a conceptual shift from potential to actual infinity and that their understanding of infinity has largely not been constructed at the object level.

Keywords: Infinity, actual infinity, potential infinity, arithmetic of infinity, prospective mathematics teacher

Introduction

Concepts play a central role in mathematics and mathematics education, shaping how mathematical ideas are constructed, understood, and communicated. Among these concepts, infinity is widely regarded as one of the most intriguing and cognitively demanding notions. Reasoning about infinity requires learners to move beyond finite experiences and intuitive forms of thinking, engaging instead with abstract entities that cannot be completed, measured, or directly perceived. Consequently, infinity poses substantial cognitive, epistemological, and pedagogical challenges (İşleyen, 2013; Yıldırım & Keşan, 2017).

At the outset, it is important to distinguish between two fundamental conceptions of infinity: potential infinity and actual infinity (Manfreda-Kolar & Hodnik-Čadež, 2012). Potential infinity refers to an endless, never-completed process, such as counting the natural numbers, whereas actual infinity denotes a completed infinite entity, such as the set of all natural numbers conceived as a whole (Manfreda-Kolar & Hodnik-Čadež, 2012; Sabery et al., 2024). Although this distinction is fundamental to modern mathematics, empirical research consistently shows that learners tend to rely

on process-based interpretations (potential infinity), which often leads to resistance toward accepting actual infinity as a legitimate mathematical object (Cihlár et al., 2015; Díaz-Espinoza et al., 2023; Krátká et al., 2021). The well-known question of whether $0.999\ldots$ equals 1, frequently discussed in the literature, illustrates the tension between potential and actual infinity (Cihlár et al., 2015; Díaz-Espinoza et al., 2023; Díaz-Espinoza et al., 2024; Juter, 2019). From the perspective of potential infinity, $0.999\ldots$ is viewed as an endlessly continuing process that never reaches completion and is therefore perceived as different from 1. From the perspective of actual infinity, however, $0.999\ldots$ is interpreted as a completed infinite sum and thus considered equal to 1.

The distinction between potential and actual infinity has been elaborated in numerous studies (Dubinsky et al., 2005; Fischbein, 2001; Hannula et al., 2006; Manfreda-Kolar & Hodnik-Čadež, 2012; Moore, 1995; Tall, 2001). Historically, this distinction can be traced back to Aristotle and was later developed by Kant (Akbulut & Akgün, 2005). The potential character of infinity is rooted in the Aristotelian tradition (Luis et al., 1991) and dominated mathematical thought until the latter part of the 19th century (Linnebo & Shapiro, 2017). Two major turning points marked the historical development of actual infinity: the work of Bernard Bolzano (1851), who introduced actual infinity to resolve paradoxical situations, and the work of Georg Cantor (1883), who formalized infinity as a mathematical object (Luis et al., 1991). Although actual infinity has contributed profoundly to the foundations of modern mathematics, it has remained conceptually controversial despite the comprehensive framework provided by Cantor's theory (Tsamir & Dreyfus, 2002). Aristotle viewed infinity as comprehensible only through potential infinity, whereas Cantor argued that potential infinity lacks meaning unless grounded in actual infinity (Cantor, 1883). Aristotle characterized actual infinity as existing at a single moment and potential infinity as unfolding over time (Moore, 1995).

From a pedagogical perspective, the distinction between potential and actual infinity is particularly important for elementary teachers, as their underlying conceptions shape how mathematical ideas are explained, represented, and discussed in the classroom. Teachers who predominantly rely on potential infinity tend to emphasize endless processes, which may lead to ambiguous or incomplete explanations of concepts such as repeating decimals, number line density, and numerical magnitude. In contrast, an understanding of actual infinity allows teachers to conceptualize infinite entities as completed objects, supporting more coherent explanations and preventing the transmission of common misconceptions. Therefore, identifying which type of infinity understanding pre-service teachers rely on is critical for evaluating their readiness to support students' conceptual development.

Theoretical Framework: Conceptual Change Theory and APOS Theory

Research grounded in conceptual change theory indicates that difficulties related to infinity do not merely stem from procedural errors but rather from deeply rooted cognitive structures developed through individuals' finite experiences. Intuitive forms of reasoning that are productive in everyday contexts become insufficient when learners are required to reason about completed infinite entities. According to the conceptual change approach, students often make sense of scientific concepts through intuitive explanations based on their everyday experiences, and they tend to resist these concepts when such explanations conflict with scientific understanding (Vosniadou, 1994; Vosniadou et al., 2008). Consequently, learning about infinity requires not the simple addition of new information but a reorganization of existing conceptual frameworks (Vosniadou & Verschaffel, 2004). Fischbein (2001) emphasizes that actual infinity is particularly resistant to cognition because it fundamentally conflicts with intuitive modes of thinking. According to conceptual change theory, as proposed by Posner et al. (1982), individuals abandon their existing conceptions only when they experience dissatisfaction with them and perceive the new conception as intelligible, plausible, and

fruitful. In this respect, infinity, due to its highly abstract nature, represents one of the most prominent cases of conflict between intuitive reasoning and formal mathematical understanding (Fischbein, 2001).

This process of conceptual restructuring can be examined in greater detail through APOS (Action–Process–Object–Schema) theory (Dubinsky & McDonald, 2001; Dubinsky et al., 2005). According to APOS theory, mathematical understanding develops through a progression from action to process, from process to object, and ultimately to schema (Dubinsky et al., 2005). At the action stage, individuals perform operations mechanically, often in response to external instructions (Dubinsky & McDonald, 2001). At the process stage, these actions are internalized, enabling mental control over the operation. At the object stage, the process is encapsulated into a coherent whole that can itself be manipulated as a mathematical entity (Dubinsky & McDonald, 2001). Finally, a schema represents an integrated cognitive structure that connects this object with related mathematical concepts (Dubinsky & McDonald, 2001; Dubinsky et al., 2005). Studies conducted in the context of infinity indicate that prospective teachers often remain at the action or process level; while they may respond correctly to certain expressions involving infinity, understanding indeterminate forms such as $\infty - \infty$ or $\infty \div \infty$ requires object-level reasoning (Hannula et al., 2006). Difficulties in achieving this process–object transition constitute one of the fundamental conceptual obstacles in understanding infinity.

Perceptions of Infinity and Arithmetic Operations Involving Infinity

Previous research has demonstrated that perceptions of infinity across different educational levels are largely grounded in potential and process-oriented interpretations (Dev & Çetin, 2023; Durak et al., 2025). Many participants, including prospective teachers, tend to conceptualize infinity as an “ongoing” and “endless” process, while object-based interpretations remain limited (Manfreda-Kolar & Hodnik-Čadež, 2012; Oflaz & Polat, 2022). This predominance of process-based reasoning is closely associated with the conceptual and procedural difficulties observed in arithmetic operations involving infinity.

The literature indicates relatively high rates of correct responses for operations such as $\infty + \infty$, $1 \cdot \infty$, and ∞^n ($n \in \mathbb{Z}^+$). In contrast, participants frequently conflate the notions of infinity, undefined, and indeterminate when dealing with indeterminate forms such as $\infty - \infty$, ∞^0 , $0 \cdot \infty$, and $\infty \div 0$ (Çelik & Akşan, 2013; Oflaz & Polat, 2022; Sırmacı & Gökkurt-Özdemir, 2016). Some studies have also reported that both students and teachers tend to treat infinity as a numerical entity, rather than as a conceptual construct with context-dependent meaning (İşleyen, 2013; Yıldırım & Keşan, 2017). Collectively, these findings underscore the strong relationship between perceptions of infinity and arithmetic competence in operations involving infinite quantities.

Epistemological and Pedagogical Dimensions of Infinity

Beyond cognitive challenges, infinity also poses significant epistemological difficulties (İşleyen, 2013; Yıldırım & Keşan, 2017). Learners frequently conflate infinity with undefined or indeterminate expressions, particularly in arithmetic contexts (Çelik & Akşan, 2013; Oflaz & Polat, 2022; Sırmacı & Gökkurt-Özdemir, 2016). These confusions reflect not only conceptual gaps but also uncertainty about the nature of mathematical existence and validity. Researchers have found that mathematics teachers’ mathematical knowledge levels are related to their students’ mathematical achievement (Hill et al., 2005). Even for elementary teachers, who may not deal with infinity directly in their curricula, knowledge of infinity is important. Many researchers have found that effective teachers know not only the content they teach directly, but also how that content fits into the larger

subject (Ball et al., 2008; Ma, 2010; Shulman, 1986). Therefore, for mathematics teachers to engage in confident and meaningful mathematical discussions in their classrooms and to support students' mathematical learning, their mathematical knowledge should not be limited to the curriculum level they teach but should also include an understanding of the underlying foundations of mathematical concepts.

In prospective mathematics teacher education programs, the concept of infinity occupies a central place in many core mathematics courses, particularly in analysis, where it is fundamental to understanding limits, convergence, and continuity. Pre-service teachers are expected to develop a sound and coherent understanding of this concept in order to make sense of advanced mathematical ideas. Although the concept of infinity is not explicitly taught in elementary grades, it is implicitly embedded in many elementary mathematics topics such as number line representations, fractions, repeating decimals, patterns, and the idea of numerical density. Elementary teachers' conceptions of infinity influence how these topics are explained, represented, and discussed in the classroom, and limited or incorrect understandings of infinity may be unintentionally transferred to students and shape their early mathematical intuitions. Accordingly, this study investigates pre-service teachers' perceptions and academic knowledge regarding the concept of infinity, which is considered a foundational concept underlying many mathematical ideas.

Previous studies consistently report persistent misconceptions about infinity across different educational levels (Bozkuş et al., 2015; Hannula et al., 2006; Oflaz & Polat, 2022). More recent research indicates that prospective teachers continue to interpret infinity predominantly in terms of potential infinity (Dev & Çetin, 2023; Durak et al., 2025). This dominant process-based view is closely linked to difficulties in arithmetic operations involving infinity. Although success rates are relatively high for operations such as $\infty + \infty$, $1 \cdot \infty$, and ∞^n , indeterminate forms such as $\infty - \infty$, ∞^0 , $0 \cdot \infty$, and $\infty \div 0$ remain problematic (Çelik & Akşan, 2013; Oflaz & Polat, 2022; Sırmacı & Gökkurt-Özdemir, 2016). Moreover, some studies show that students and even teachers treat infinity as a number (İşleyen, 2013; Yıldırım & Keşan, 2017).

Overall, the literature highlights a strong relationship between infinity conceptions and performance in arithmetic operations involving infinity. Accordingly, the present study aims to examine prospective mathematics teachers' perceptions of infinity and their academic knowledge of arithmetic operations involving ∞ , 0, 1, and π . The study is guided by the following research questions:

- How do prospective mathematics teachers perceive infinity?
- How well do prospective mathematics teachers solve problems involving operations with ∞ , 0, 1, and π ?

Methods

In this study, a concurrent nested mixed-method design was used—quantitative and qualitative data were collected together, but emphasis was given to quantitative data (Baki & Gökçek, 2012; Creswell, 2014; Yıldırım & Şimşek, 2016).

Study Group of the Research

The study group of the research consisted of 100 volunteer prospective teachers studying in the third and fourth grades of the Department of Mathematics Education at a state university in the Aegean Region of Türkiye in the spring semester of the 2024-2025 academic year. These prospective teachers are enrolled in a four-year undergraduate teacher education program in Türkiye. At the time

of the study, 54 of the participants were in their third year, and 46 were in their fourth year of this university program. Upon graduation, these prospective teachers are qualified to teach mathematics at the lower secondary level (middle school), specifically to students in Grades 5 through 8 (approximately ages 10–14) in Türkiye. The study group of the research was determined by a criterion sampling technique, one of the purposeful sampling methods (Patton, 2001). The criterion used was that participants successfully completed the courses Analysis I, Analysis II, and Analysis III, which are the courses dealing with infinity and different variations of limit. In addition, since these prospective teachers were in the final years of their programs, they had completed most of the mathematics and mathematics education courses required by the curriculum. Therefore, it can be argued that these preservice teachers had many opportunities to learn about infinity.

In the curriculum of the mathematics teacher education program in Türkiye, prospective teachers are required to take a series of Analysis courses beginning in their first semester, in accordance with the course content prescribed by the Council of Higher Education (Council of Higher Education [CoHE], 2018). These courses, titled Analysis I, Analysis II, and Analysis III, are sequentially offered across consecutive semesters. In Türkiye, prospective teachers are first introduced to the concept of limits—particularly those involving multivariable functions—within these Analysis courses. Alongside fundamental concepts, the curriculum includes topics such as the limits of single-variable functions at a point and the limits of two-variable functions, which are treated as a continuation of the previous topics in successive courses. In the Analysis courses, prospective teachers study topics that address the concept of infinity and related arithmetic operations within various limit contexts. Table 1 presents detailed information regarding the content of the Analysis courses.

Table 1

Contents of Analysis I, Analysis II, and Analysis III courses (CoHE, 2018)

Analysis I	Analysis II	Analysis III
Sets and number systems; relations, types of functions, exponential functions, and logarithmic functions; limits, continuity concepts, and their applications; derivatives, applications of derivatives, and graphing (p. 4).	Trigonometric functions, trigonometric relations, solutions of trigonometric equations; complex numbers and their properties; Riemann sum, definite integral, indefinite integral, methods of integration, applications of integration, improper integrals, series and convergence tests (p. 6).	Multivariable functions; topology of $\mathbb{R}^n$, limit, continuity, function sequences and series; derivative, directional derivative, partial derivative, geometric interpretation of partial derivative, higher-order derivatives and chain rule (p. 7).

Data Collection Tool and Data Collection Process of the Research

The data of the study were collected using an online form consisting of 25 questions in total, one open-ended and 24 multiple-choice problems with basic arithmetic operations related to infinity, compiled and expanded from mathematics and mathematics education literature (Aşık, 2010; Çelik & Akşan, 2013; Oflaz & Polat, 2022; Sergeyev, 2003; Sırmacı & Gökkurt-Özdemir, 2016). The open-ended question aims to determine the perceptions of infinity of prospective mathematics teachers. This question is “*What comes to your mind when you hear the word infinity? Explain*” question. Multiple-choice questions are aimed at determining the academic knowledge of prospective teachers regarding the arithmetic of infinity. Multiple-choice questions consist of four parts. The first part consists of the operations involving ∞ and ∞ , the second part consists of the operations involving ∞ and 1, the third part consists of the operations involving ∞ and 0, and the fourth part consists of

the operations involving ∞ and π . Each part consists of six questions. Of the 24 multiple-choice questions, 10 have the correct answer ∞ , seven have the correct answer indeterminate, five have the correct answer 0, and two have the correct answer $-\infty$. Participants were given 25 minutes to complete the form. The form is presented in Appendix-1.

Participants accessed and answered the questions in class during the spring semester of the 2024-2025 academic year. Prospective teachers accessed the form via the link sent and submitted their answers individually within the set 25-minute period.

Data Analysis

The responses to the open-ended question in the online form were analyzed using qualitative descriptive analysis. For this purpose, four codes were used: *actual infinity*, *potential infinity*, *both actual and potential infinity*, and *meaningless*. The responses of prospective teachers who perceived infinity as an object or a completed entity were coded as *actual infinity*, the responses of prospective teachers who perceived it as an ever-continuing process were coded as *potential infinity*, and the responses referring to both the process and the object were coded as *both actual and potential infinity*. Answers other than these, which have nothing to do with infinity, were coded as *meaningless*. The frequencies and percentages of the codes obtained are reflected in Table 1.

Each correct answer to multiple-choice questions was scored as one point, and each wrong or blank answer was scored as zero points. These data were analyzed with quantitative descriptive statistics through the aforementioned spreadsheet and statistics program. The arithmetic mean scores obtained from the answers given to 24 multiple-choice questions are presented and interpreted in Table 2. In addition, the arithmetic mean scores of the responses to the operations ∞ and ∞ , ∞ and 1, ∞ and 0, and ∞ and π are also reflected and interpreted in this table according to separate class levels. In addition, minimum and maximum scores are given and interpreted as additional information. The frequency and percentage values of correct answers given to each multiple-choice question are presented and interpreted in Table 3 according to class levels in four parts.

Validity and Reliability Studies

The researchers tried to ensure and increase the internal and external validity and reliability (Büyükoztürk et al., 2016; Creswell, 2014; Karasar, 2014; Yıldırım & Şimşek, 2016) of the study. For the content validity of the form, expert opinions (Büyükoztürk et al., 2016; Creswell, 2014; Hair et al., 2014) were taken from a professor from the departments of mathematics and mathematics education. In this context, opinions were received from experts on whether the form was intended to determine prospective mathematics teachers' perceptions of infinity and their academic knowledge about the arithmetic of infinity.

In line with these views, the number of multiple-choice questions was reduced from 30 to 24. Questions $-\infty - \infty$, $(+\infty) - (-\infty)$, $(-\infty) \cdot (-\infty)$, $\pi - (-\infty)$, $(-1)^\infty$, and $n \in \mathbb{Z}^+$ while ∞^n were removed from the form in order to both reduce the number of questions and eliminate similar questions. Thus, the total number of questions in the form became 25, including the open-ended question. In addition, the time to answer the form has been reduced from 30 minutes to 25 minutes. Data were collected in the participants' natural environment, preventing them from being exposed to any factors.

Qualitative descriptive analysis of the responses to the open-ended question was conducted separately by both authors. For this purpose, two codes determined within the theoretical framework of the research were used. These codes are the codes of *actual infinity* and *potential infinity*.

Thus, out of 100 codes, 90 were found to be compatible and 10 were found to be incompatible. The agreement percentage between the coders was calculated as $\frac{90}{90+10} = 90\%$ using the “*reliability = $\frac{\text{number of agreements}}{\text{total number of agreements+disagreements}}$* ” formula of Miles and Huberman (1994, p. 64). It was determined that eight of the incompatible responses contained expressions regarding perceptions of both actual and potential infinity. The answers of the two prospective teachers did not fall into any of these categories. In order to resolve the incompatibilities, *both the actual and potential infinity* code and the *meaningless* code were included in the coding, and the frequency and percentage values of these four codes are reflected in Table 1. For example, below is a sample statement from a prospective teacher that evokes both potential and actual infinity.

When I think of infinity, a concept that has no end or ending comes to mind. In mathematics, numbers continue forever; in the universe, time and space can be thought of as infinite. It expresses a magnitude or continuity that has no end.

Although authors initially disagreed by attempting to determine the dominant side in this statement, they resolved their disagreements by including such statements in both the actual and potential infinity categories. In addition, participant responses to these four different codes are presented to the reader with verbatim quotes (Guba & Lincoln, 1981).

Quantitative descriptive statistics of the responses to multiple-choice questions were made by one of the authors using a spreadsheet program and by the other author using a statistics program, and the results were compared. No disagreements were found in these results.

Finally, the open-ended and multiple-choice questions in the data collection tool are given in the appendix, and the online and classroom data collection process is presented to the reader in detail. The four codes and examples in the qualitative data analysis and the process of reaching the scores and values in the quantitative data analysis are also described in the same way.

Findings

In this section, the findings of the research questions “*How do prospective mathematics teachers perceive infinity?*” and “*How well do prospective mathematics teachers solve problems involving operations with ∞ , 0, 1, and π ?*” are presented, respectively.

Perceptions of Infinity of Prospective Mathematics Teachers

In the first stage of the study, the aim was to reveal prospective teachers’ perceptions of infinity. For this purpose, prospective teachers were asked the question, “*What comes to your mind when you hear the word infinity? Explain.*” All prospective teachers have a perception about infinity; there is no prospective teacher who left this open-ended question unanswered.

An analysis of their responses indicated that the answers could be grouped into four main categories. Table 2 presents detailed information regarding prospective teachers’ perceptions of the concept of infinity.

Table 2*Perceptions of Infinity by Prospective Teachers*

Perception of Infinity	Grade 3		Grade 4		Grade 3 and 4	
	f	%	f	%	f	%
Actual infinity	23	42.59	14	30.44	37	37.0
Potential infinity	26	48.15	27	58.70	53	53.0
Both actual and potential infinity	3	5.56	5	10.87	8	8.0
Meaningless	2	3.70	0	0	2	2.0
Total	54	100	46	100	100	100

Note. Percentage values have been rounded to the hundredths.

As seen in Table 2, prospective mathematics teachers' perceptions of infinity are more frequently associated with potential infinity (53%). However, perceptions related to actual infinity (37%) also have a considerable percentage and ratio. The percentage of prospective teachers who use expressions related to both actual and potential infinity is 8%. Only two prospective teachers (2%) gave meaningless answers. No prospective teachers were encountered who left the question blank. The order of frequencies and percentages of these codes (potential, actual, both actual and potential, meaningless) does not change in the perceptions of both third and fourth-grade prospective teachers. However, in the perceptions of third-grade prospective teachers, the percentages of actual infinity and potential infinity are closer to each other. Accordingly, these findings indicate that the majority of prospective teachers perceive infinity intuitively as an unending process. It can be stated that a substantial proportion of the prospective teachers are able to construct the concept of infinity as a mathematical object. This suggests that, at least partially, a conceptual change has occurred. Moreover, the small number of prospective teachers who adopted both types of infinity demonstrates their ability to transform intuition-based views of infinite processes into mathematically structured objects, indicating a more advanced level of conceptual change.

Prospective teachers with perceptions related to potential infinity mostly wrote answers that infinity is a never-ending, beginningless, and endless, ever-continuing, increasing, or multiplying process. Verbatim quotes from the answers of seven prospective teachers with perceptions related to potential infinity are given below.

- “A situation that has no end, never ends, is continuous and ongoing.”
- “It means that it has no end, never ends, and is constantly continuing.”
- “Unending, continuing, constantly increasing.”
- “A situation that has no beginning or end, that is constantly advancing or regressing.”
- “Having no end, always continuing.”
- “The absence of any end means that it is constantly continuing.”
- “A concept that is unending, with no beginning or end.”

Prospective teachers who have perceptions related to actual infinity often expressed infinity as an infinite set or object. In addition, some of these prospective teachers wrote statements about the infinity of the world, space, or the universe. Verbatim quotes from the responses of six prospective teachers who have perceptions related to actual infinity are given below.

- “An infinite set, a set of solutions that can take any value.”
- “Infinity reminds me of the universe.”
- “Outer space.”

“A collection of objects that has no end comes to mind. In other words, a set that contains every number I can think of.”

“A collection of objects that has no end.”

“An endless collection of objects.”

Expressions related to both actual and potential infinity were encountered in the answers of eight prospective teachers. These prospective teachers have included both process- and object-oriented expressions separately. Verbatim quotes from the answers of four prospective teachers who wrote answers related to both actual and potential infinity are given below.

“When I think of infinity, a concept that has no end or ending comes to mind. In mathematics, numbers continue forever; in the universe, time and space can be thought of as infinite. It expresses a magnitude or continuity that has no end.”

“When I think of infinity, what comes to my mind is limitless, endless, the infinity of the universe, something that has no beginning or end.”

“Endless, space, right.”

“Endless, limitless, the universe that has no beginning or end.”

Both prospective teachers coded as “meaningless” wrote the answer “indeterminate”. These prospective teachers think that the concepts of infinity and indeterminate express the same thing.

Academic Knowledge of Prospective Mathematics Teachers Regarding the Arithmetic of Infinity

Following the examination of prospective teachers’ perceptions of the concept of infinity, their responses to the 24 multiple-choice questions were analyzed. The results are presented in Table 3.

Table 3

Descriptive Statistics Values for the Arithmetic of Infinity

	Part	Number of Questions	N_3	N_4	$N_{3 \text{ and } 4}$	$\overline{X_3}$	$\overline{X_4}$	$\overline{X_{3 \text{ and } 4}}$
Part 1	∞ and ∞	6	54	46	100	3.32	4.87	4.03
Part 2	∞ and 1	6	54	46	100	4.33	5.61	4.92
Part 3	∞ and 0	6	54	46	100	1.85	3.17	2.46
Part 4	∞ and π	6	54	46	100	4.41	5.24	4.79
Total		24	54	46	100	13.91	18.89	16.20

Note. Arithmetic mean values have been rounded to the hundredths.

As seen in Table 3, prospective mathematics teachers answered, on average, 16.20 out of 24 items correctly, corresponding to a success rate of approximately 67.5%. Considering that 12 correct responses (50%) represent the midpoint of the test, this result indicates that the prospective teachers’ academic knowledge of the arithmetic of infinity is above the midpoint level. Fourth-grade prospective teachers are more successful than third-grade prospective teachers in the arithmetic of infinity ($\overline{X_4}=18.89>\overline{X_3}=13.91$). Prospective mathematics teachers were least in the operations of

∞ and 0. This was especially true for third-grade prospective teachers, but also held true for fourth-grade prospective teachers. The operations that prospective mathematics teachers are most successful at are ∞ and 1. According to the grade levels, fourth-grade prospective teachers were most successful in operations with ∞ and 1, while third-grade prospective teachers were most successful in operations with ∞ and π . As additional information, the minimum score for third-grade prospective teachers is six, and the maximum score is 21, while the minimum score for fourth-grade prospective teachers is eight, and the maximum score is 22. In other words, the prospective mathematics teacher with the lowest number of correct answers was able to answer six questions out of 24 multiple-choice questions correctly, while the prospective teacher with the highest number of correct answers was able to answer 22 questions correctly.

The frequency and percentage values of the correct answers given by the prospective mathematics teachers to each multiple-choice question are presented in Table 4 in parts according to their grade levels.

Table 4

Frequency and Percentage Values for the Arithmetic of Infinity

Part	Question No	Question	Correct Answer					
			Grade 3 ($N_3 = 54$)		Grade 4 ($N_4 = 46$)		Grade 3 and 4 ($N_{3 \text{ and } 4} = 100$)	
			f	%	f	%	f	%
Part 1 ∞ and ∞	Question 1	$\infty + \infty$	45	83.33	45	97.83	90	90.0
	Question 2	$\infty - \infty$	23	42.59	38	82.61	61	61.0
	Question 3	$\infty \cdot \infty$	27	50.00	42	91.30	69	69.0
	Question 4	$\infty \div \infty$	34	62.96	38	82.61	72	72.0
	Question 5	$(-\infty) \div (-\infty)$	33	61.11	39	84.78	72	72.0
	Question 6	∞^∞	17	31.48	22	47.83	39	39.0
Part 2 ∞ and 1	Question 7	$\infty \div 1$	51	94.44	46	100.0	97	97.0
	Question 8	$1 \div \infty$	37	68.52	44	95.65	81	81.0
	Question 9	∞^1	48	88.89	44	95.65	92	92.0
	Question 10	1^∞	11	20.37	35	76.09	46	46.0
	Question 11	∞^{-1}	34	62.96	43	93.48	77	77.0
	Question 12	$1 \cdot \infty$	53	98.15	46	100.0	99	99.0
Part 3 ∞ and 0	Question 13	∞^0	14	25.93	35	76.09	49	49.0
	Question 14	$0 \cdot \infty$	9	16.67	33	71.74	42	42.0
	Question 15	0^∞	30	55.56	14	30.43	44	44.0
	Question 16	$0 \cdot (-\infty)$	12	22.22	31	67.39	43	43.0
	Question 17	$0 \div \infty$	31	57.41	30	65.22	61	61.0
	Question 18	$\infty \div 0$	4	7.41	3	6.52	7	7.0
Part 4 ∞ and π	Question 19	$\infty + \pi$	47	87.04	46	100.0	93	93.0
	Question 20	$\infty \div \pi$	38	70.37	38	82.61	76	76.0
	Question 21	$\pi - \infty$	35	64.81	36	78.26	71	71.0
	Question 22	$\pi \cdot (-\infty)$	45	83.33	43	93.48	88	88.0
	Question 23	π^∞	35	64.81	36	78.26	71	71.0
	Question 24	$\pi^{-\infty}$	38	70.37	42	91.30	80	80.0
Total			751	57.95	869	78.71	1620	67.5

Note. Percentage values have been rounded to the hundredths.

When considered on a question-by-question basis, the question that prospective mathematics teachers are least successful at (7%) is the $\infty \div 0$ question. In this question, the majority of the prospective teachers (80%) chose option undefined, 12% of the remaining ones chose option indeterminate, 7% options ∞ , and 1% option 0. The question that prospective mathematics teachers are most successful at (99%) is question $1 \cdot \infty$. All fourth-grade prospective teachers (100%) and almost all third-grade prospective teachers (98.15%) gave the correct answer to this question.

When considered on the part basis, the question that prospective mathematics teachers are least successful in regarding ∞ and ∞ operations is ∞^∞ . Only 39% of the prospective mathematics teachers chose the correct answer option ∞ , 48% chose option indeterminate, 11% chose option undefined, and the remaining 2% chose option 0. In this question, undefined and indeterminate were the distractors for the prospective teachers. The question that prospective mathematics teachers are most successful at (90%) in ∞ and ∞ operations is $\infty + \infty$. More than half of the fourth-grade prospective teachers were able to choose the correct answer for all other ∞ and ∞ operations. The majority of third-grade prospective teachers could not mark the correct answer in the $\infty - \infty$ question in addition to the ∞^∞ question. The percentage of correct answers is 42.59. The distractor options for this question were 0 and ∞ . 13 third-grade prospective teachers (24.07%) chose the answer 0, and again 13 third-grade prospective teachers (24.07%) chose the answer ∞ . Half of the third-grade prospective teachers were able to answer the question $\infty \cdot \infty$ correctly. $\infty \div \infty$ and $(-\infty) \div (-\infty)$ indeterminates are the indeterminates in which pre-service elementary mathematics teachers have the highest percentage (72%) among 1^∞ , ∞^0 , $0 \cdot \infty$, $0 \cdot (-\infty)$, $\infty - \infty$, $\infty \div \infty$, $(-\infty) \div (-\infty)$ indeterminates, that is, they are the most successful.

The highest percentage of correct answers for ∞ and 1 operations is the $1 \cdot \infty$ question mentioned above, which has the highest percentage (99%) among all questions. The question with the lowest correct percentage (46%) in the operations ∞ and 1 is the only question in this section with an indeterminate answer, namely 1^∞ . There is no question that more than half of the fourth-grade prospective teachers answered incorrectly. However, the correct percentage of third-grade prospective teachers for question 1^∞ was 20.37%, while the incorrect percentage was 79.63%. 38 third-grade prospective teachers (70.37%) marked the answer 1 in this indeterminate question. The majority of third-grade prospective teachers were able to mark the correct options in other ∞ and 1 operations except for this question.

In the operations of ∞ and 0, the majority of the prospective mathematics teachers could not mark the correct answer in the questions of ∞^0 , $0 \cdot \infty$, 0^∞ , $0 \cdot (-\infty)$, and $\infty \div 0$, except for the question of $0 \div \infty$. The lowest percentage of correct answers for ∞ and 0 operations is the $\infty \div 0$ question mentioned above, which has the lowest percentage (7%) among all questions. The majority of fourth-grade prospective teachers could not mark the correct answer in the 0^∞ and $\infty \div 0$ questions, and the majority of third-grade prospective teachers could not mark the correct answer in the ∞^0 , $0 \cdot \infty$, $0 \cdot (-\infty)$, and $0 \div \infty$ questions. In question ∞^0 , 1, in the question $0 \cdot \infty$, 0, in the question 0^∞ , indeterminate, in the question $0 \cdot (-\infty)$, 0, and in the question $\infty \div 0$, undefined and indeterminate were the distractor options.

In the operations of ∞ and π , the majority of prospective mathematics teachers were able to mark the correct answer in all questions. The majority of both third-grade and fourth-grade prospective teachers were able to mark the correct options in all questions. The highest correct percentage (93%) belongs to the question $\infty + \pi$ (93%), while the lowest correct percentage (71%) belongs to the questions $\pi - \infty$ and π^∞ .

Considering the grade level, fourth-grade prospective teachers were more successful than third-grade prospective teachers in all questions related to the operations ∞ and ∞ , ∞ and 1, and ∞ and π . In the questions about the operations ∞ and 0, the correct percentage of third-grade prospective teachers is higher than the fourth-grade prospective teachers only in the questions 0^∞ and $\infty \div 0$. In the other four questions, it is seen that the correct percentage of fourth-grade prospective teachers is higher than the third-grade prospective teachers.

Discussion

The first results of the study are related to the perceptions of infinity of prospective mathematics teachers. It was determined that the perceptions of infinity of prospective mathematics teachers were more frequently related to potential infinity (Cihlár et al., 2015; Díaz-Espinoza et al., 2023; Díaz-Espinoza et al., 2024; Krátká et al., 2021). In their study aimed at improving high school mathematics teachers' conceptions of infinity, Díaz-Espinoza et al. (2023) reported that the teachers predominantly held a potential conception of infinity prior to the intervention, and that they showed resistance to changing this conception even after the instructional activities. This finding revealed that the potential conception of infinity represents a robust and persistent cognitive image. Similarly, İşleyen (2013) found in his study that the understanding of potential infinity was dominant among high school students. Dev and Çetin (2023) determined in their study that all of the metaphors produced by high school students about infinity were related to potential infinity.

Although the perceptions related to potential infinity were more intense in our study, the perceptions related to actual infinity were also considerable. Especially in the perceptions of third-grade prospective teachers, the percentages of actual infinity and potential infinity are closer to each other. In their study examining prospective teachers' understanding of the concept of infinity and the factors affecting these understandings, Manfreda-Kolar and Hodnik-Čadež (2012) revealed that prospective teachers' justifications for solving tasks related to infinity were based on both actual and potential infinity. Oflaz and Polat (2022) revealed that students of the mathematics department and mathematics teaching department evaluate infinity as operational, intuitive, potential, and actual infinity, and that they have such understandings about infinity. In order to conceptualize infinity and internalize the arithmetic of infinity, both the concepts of actual and potential infinity are needed. Therefore, it can be suggested that this distinction should be conveyed to prospective teachers along with its historical processes in teacher preparation programs.

Prospective mathematics teachers who have perceptions related to potential infinity mostly (53%) described infinity as a process that has no beginning or end, never ends, continues continuously, and increases or decreases continuously. Prospective teachers (37%) who have perceptions related to actual infinity also mostly described infinity as an infinite set or object. Some have also written statements about the infinity of the world, space, or the universe. In a similar study, Maria et al. (2010), who investigated whether teachers' perceptions of infinity were a process or an object, found that approximately two-thirds of the teachers (72.1%) defined infinity as a process, and the rest (27.9%) defined it as an object. Teachers who defined infinity as a process used expressions such as "it goes on forever," "it's a process that never ends," "it has no beginning and no end...always follows another number," "keeps going and increasing," while teachers who defined infinity as an object used expressions such as "it is an infinite whole," "it is something countless," "it is a set with unlimited elements," and "it is an undefined set" (Maria et al., 2010, p. 1775). In these respects, the parallel aspects of the two studies are striking. In the studies of Oflaz and Polat (2022), while some of the mathematics and mathematics teaching students emphasize the ever-increasing aspect of infinity, some of them have an understanding of infinity as discussed by Cantor in the context of countable and uncountable infinite sets. Unlike the results of these studies, only one of

the 13 mathematics teachers in the study by Sırmacı and Gökkurt-Özdemir (2016) expressed infinity as a process.

In the answers of eight prospective teachers, expressions related to both actual and potential infinity were encountered. These prospective teachers used expressions in which they described infinity as both a process and an object. It is also interesting that only two prospective teachers' perceptions of infinity were evaluated in the meaningless category. 98 prospective teachers were able to give meaningful answers, while two prospective teachers gave a meaningless answer by writing "indeterminate." Çelik and Akşan (2013) also asked prospective mathematics teachers the question "*Explain the concept of infinity in your own words*" and the answers of 16 out of 83 participants were evaluated in the category of incorrect explanation because they generally referred to undefined and indeterminate (pp. 173-174). Although there is a difference in the frequencies of the prospective teachers who gave meaningless and incorrect answers in the two studies, their references to indeterminate may be an example of their misperceptions about infinity. These misconceptions may affect the learning of infinity-related topics (e.g. limit and series) and even courses (e.g., Analysis) in teacher education. It may be beneficial for instructors of courses involving infinity in teacher education to be aware of these misconceptions for their teaching.

From the perspective of conceptual change theory, findings indicate that more than half of the prospective teachers (53%) perceived infinity in terms of potential infinity, describing it as an "endless, continuously ongoing process." This result is consistent with a substantial body of literature suggesting that human cognition exhibits a natural tendency toward potential infinity (Dubinsky et al., 2005; Fischbein, 2001; Hannula et al., 2006). This tendency implies that prospective teachers have not yet experienced dissatisfaction with their existing intuitive conceptions and, therefore, have not reconstructed infinity as a completed mathematical object. The fact that 37% of the prospective teachers expressed infinity in terms of actual infinity suggests that conceptual change has partially occurred for some individuals. However, the finding that only 8% simultaneously considered both potential and actual infinity reveals that relatively few prospective teachers have developed a coherent and integrated cognitive structure linking these two conceptions. According to conceptual change theory, this indicates that most prospective teachers remain unable to abandon intuition-based reasoning grounded in potential infinity and continue to interpret actual infinity within this intuitive framework (Posner et al., 1982; Vosniadou, 1994; Vosniadou et al., 2008).

As shown in Table 3, the prospective mathematics teachers answered an average of 16.20 items correctly on the 24-item arithmetic of infinity test. This score exceeds the midpoint of the test, which corresponds to 12 correct responses (50%), and represents an overall success rate of approximately 67.5%. Accordingly, this finding indicates that the prospective teachers' academic knowledge of the arithmetic of infinity is above the midpoint level of the assessment. When examined by grade level, fourth-year prospective teachers demonstrated higher performance than third-year prospective teachers. This difference may be attributed to the fourth-year students' greater exposure to advanced mathematical content, theoretical coursework, and pedagogical training throughout the teacher education program. In a similar vein, Hannula et al. (2006) reported that, in parallel with age-related cognitive development, learners tend to develop more appropriate and structured conceptions of infinity. Beyond overall performance and grade-level differences, distinct patterns were also observed across different types of infinity operations. Specifically, prospective teachers showed lower performance on operations involving ∞ and 0 compared to their performance on other infinity-related operations. In contrast, operations involving ∞ and 1 were those in which prospective teachers were most successful. This pattern suggests that prospective teachers may tend to perceive infinity as an invariant quantity, which leads to conceptual difficulties particularly in operations involving 0, whereas operations involving 1 may be approached using more intuitive and rule-based reasoning. These findings highlight the need to address conceptual

limitations related to the arithmetic of infinity more explicitly within mathematics teacher education programs.

A detailed examination of the responses to the questions revealed that prospective mathematics teachers were most successful in operations involving $\infty + \infty$ and $\infty \div \infty$. In particular, the majority of both third- and fourth-year prospective teachers correctly identified $\infty + \infty$ as ∞ , with higher success rates observed among fourth-year students. Similarly, $\infty \div \infty$ and $(-\infty) \div (-\infty)$ were the indeterminate forms with the highest rates of correct responses. In contrast, operations such as $\infty - \infty$, ∞^0 , 1^∞ , $0 \cdot \infty$, and $0 \cdot (-\infty)$ were found to be the most challenging. In these cases, many prospective teachers selected distractor options such as 0, 1, ∞ , or “undefined,” indicating persistent conceptual confusion. Third-year prospective teachers were generally less successful than fourth-year students, particularly in recognizing indeterminate forms. For operations involving ∞ and 1, almost all prospective teachers correctly answered $1 \cdot \infty$, whereas 1^∞ was among the least correctly answered questions, often being interpreted as 1. In operations involving ∞ and 0, only $0 \div \infty$ yielded relatively high correct response rates, while $\infty \div 0$ was frequently misinterpreted as undefined. These results suggest that 0 and 1 function as strong distractors alongside ∞ , indeterminate, and undefined in the arithmetic of infinity. As a contribution beyond the existing literature, this study also examined operations involving π and infinity. The results showed that prospective teachers generally did not experience difficulty with these operations. Overall, the findings reveal that prospective teachers frequently confuse the concepts of infinity, indeterminate, undefined, 0, and 1, particularly in operations requiring object-level reasoning about infinity. Taken together, these results are consistent with previous studies reporting similar misconceptions and difficulties related to the arithmetic of infinity and indeterminate forms (Çelik & Akşan, 2013; İşleyen, 2013; Oflaz & Polat, 2022; Sırmacı & Gökkurt-Özdemir, 2016; Yıldırım & Keşan, 2017).

When prospective teachers’ achievement in arithmetic operations involving infinity is examined within the framework of APOS theory, a clear pattern emerges. The high success rates observed in operations such as $\infty + \infty$, $\infty \cdot 1$, and $\infty + \pi$ suggest that prospective teachers tend to approach these operations primarily at the action or process level. Because such operations can be intuitively generalized as “infinity is already very large,” they can often be answered correctly without requiring a deep level of objectification. In contrast, the substantial decline in success rates for operations such as $\infty - \infty$, $0 \cdot \infty$, $\infty \div \infty$, $\infty \div 0$, and ∞^∞ indicates that prospective teachers experience considerable difficulty in conceptualizing infinity at the object level. In particular, the fact that the correct response rate for the $\infty \div 0$ item was only 7% reveals that the distinctions among infinity, indeterminacy, and undefinedness have not been adequately constructed at the object stage of APOS theory. According to APOS theory, the transformation of a process into an object requires individuals to conceive the process as a coherent whole and to perform mental operations on it (Dubinsky & McDonald, 2001). The findings suggest that most prospective teachers continue to interpret infinity as an “ongoing process,” which leads to cognitive conflict when confronting indeterminate forms such as $\infty - \infty$ or $0 \cdot \infty$. This helps explain why the options “indeterminate” and “undefined” function as strong distractors, particularly in items such as ∞^∞ and 1^∞ .

Finally, the findings revealed that prospective teachers’ conceptions of infinity are directly reflected in their performance in the arithmetic of infinity. The results indicate that most prospective teachers predominantly hold a potential infinity perspective, conceptualizing infinity intuitively as an endless, continuously increasing process. Within this framework, they were relatively successful in operations whose results are infinite, whereas they experienced considerable difficulty in situations involving indeterminate forms. In other words, prospective teachers tended to perform well in contexts that can be addressed through potential infinity reasoning but struggled in cases that require an object-level understanding of actual infinity. This finding demonstrates that prospective teachers’ perceptions of infinity are closely associated with their arithmetic reasoning involving

infinity. This central result is in direct alignment with the theoretical framework of the study. When conceptual change theory and APOS theory are considered together, the findings suggest that prospective teachers' learning process regarding the concept of infinity remains largely incomplete. Intuitive understandings grounded in potential infinity continue to dominate among a substantial proportion of prospective teachers. Unless such intuitive structures give rise to sufficient cognitive conflict, as proposed by conceptual change theory, the construction of actual infinity as a mathematical object becomes difficult. In particular, the observed performance gap between operations yielding infinite results (e.g., $\infty + 1$, $\infty \cdot \pi$) and operations yielding indeterminate results (e.g., $\infty - \infty$, $0 \cdot \infty$, $\infty \div \infty$) clearly indicates that prospective teachers succeed when intuitive generalizations are sufficient but encounter difficulties in situations that require objectification. From the perspective of APOS theory, this pattern points to an incomplete transition from process to object, while from a conceptual change perspective, it suggests that prospective teachers have not yet experienced an adequate level of dissatisfaction with their existing conceptions.

Conclusion and Recommendations

This study examined prospective mathematics teachers' perceptions of infinity and how these perceptions are reflected in their understanding of the arithmetic of infinity. The findings indicate that the majority of prospective teachers predominantly conceptualize infinity in terms of potential infinity, while fewer demonstrate an understanding of actual infinity, and only a small number are able to coordinate both perspectives. Given that intuitive, process-based views of infinity may lead to systematic errors in abstract mathematical contexts, these results highlight the need for instructional practices that support prospective teachers' conceptual transition toward actual infinity. In particular, teacher education programs should incorporate learning experiences that explicitly promote the objectification of infinity and support its construction at the object level.

Another important finding of the study is that prospective teachers' success in operations involving infinity varies considerably depending on the type of operation. While participants generally performed well on operations whose outcomes are infinite, they experienced substantial difficulty with operations that result in indeterminate forms (e.g., $\infty - \infty$, $0 \cdot \infty$, $\infty \div \infty$). This pattern suggests that prospective teachers' understanding of infinity often remains at the process level, which is sufficient for handling intuitive generalizations but inadequate for situations requiring object-level reasoning. Given that limit situations and indeterminate forms play a central role across many areas of mathematics, instructional efforts should aim to ensure that these concepts are understood not merely procedurally but as coherent mathematical objects. In this respect, the adoption of APOS-based instructional approaches is strongly recommended for teaching the arithmetic of infinity. Instruction should be designed to support prospective teachers' progression from action and process levels to object-level reasoning, particularly in contexts involving indeterminate forms. Emphasizing infinity as a completed mathematical entity and engaging learners in reasoning about its properties may help reduce persistent misconceptions and foster deeper conceptual understanding.

Finally, the findings demonstrate a clear relationship between prospective teachers' conceptions of infinity and their performance in arithmetic operations involving infinity. The dominance of potential infinity appears to be a key factor underlying many of the observed difficulties. This central result is consistent with the theoretical framework of the study. When considered together, conceptual change theory and APOS theory suggest that prospective teachers' learning of the concept of infinity remains largely incomplete, as the intuitive, process-based understanding has not yet been sufficiently challenged to trigger the cognitive conflict necessary for restructuring. Future research may further investigate this relationship using alternative theoretical perspectives and research designs. In particular, experimental and longitudinal studies focusing on

instructional interventions aimed at fostering an understanding of actual infinity could provide valuable insights for improving teacher education practices.

Limitations

Despite its contributions, this study has several limitations that should be considered when interpreting the findings. First, the study focused on prospective mathematics teachers in their third and fourth years of study at a single state university. Therefore, the findings may not be directly generalizable to other contexts, institutions, or educational levels.

Second, the qualitative component of the study was limited to responses obtained from a single open-ended question, while the quantitative findings were based on a structured multiple-choice instrument developed for this research. Although this design allowed participants to express their reasoning with minimal constraints, it may not have fully captured the depth and complexity of their conceptual understandings. Future research could replicate the study using different research designs, such as in-depth interviews, think-aloud protocols, or classroom-based interventions, to gain richer insights into prospective teachers' reasoning about infinity.

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Appendix-1: Form

Informed Consent and Assent Guide: Dear prospective mathematics teachers, this form has been prepared to examine your perceptions of infinity and your academic knowledge about the arithmetic of infinity. The data obtained from this form will be used in the scientific study titled “<i>Prospective mathematics teachers’ perceptions of infinity and their academic knowledge about the arithmetic of infinity</i>”, with the right to change the name, and your identity will remain confidential. This form consists of one open-ended and 24 multiple-choice questions. This form will be applied to you online in a classroom environment and you will be given 25 minutes for this. Your participation in this research is voluntary. You can participate in the research, refuse to participate, or withdraw from the research at any time after participation. When you participate in the research, you will not face any risks during the research. You will never face any negative consequences if you do not participate in the research or withdraw from the research. In addition, if you withdraw from the research, your data will be removed from the scope of the research. Before the research, you need to have detailed information about the purpose, scope, duration of the research, and your participation rights. You can ask the researcher any unclear issues and questions you want before or during the research. If you decide to participate, do not be under anyone's influence or pressure while answering the form. “I have read and understood the Informed Consent and Assent Guide.” ☐ I want to participate in the research of my own free will. ☐ I do not want to participate in the research. 1. What comes to your mind when you hear the word infinity? Explain.		
Questions	Correct Answer	Brief Explanation
$\infty + \infty$	∞	Adding two unbounded quantities results in an unbounded (infinite) quantity. Example: $\lim_{x \rightarrow \infty} (x + x) = \infty$.
$\infty - \infty$	Indeterminate	The difference between two unbounded quantities can take any value depending on their growth rates. Example: $\lim_{x \rightarrow \infty} (x - x) = 0$ and $\lim_{x \rightarrow \infty} (2x - x) = \infty$.
$\infty \cdot \infty$	∞	The product of two unbounded positive quantities diverges to infinity. Example: $\lim_{x \rightarrow \infty} (x \cdot x) = \infty$.
$\infty \div \infty$	Indeterminate	The ratio of two unbounded quantities depends on their rates of growth. Example: $\lim_{x \rightarrow \infty} \left(\frac{x}{x}\right) = 1$, $\lim_{x \rightarrow \infty} \left(\frac{x^2}{x}\right) = \infty$, $\lim_{x \rightarrow \infty} \left(\frac{x}{x^2}\right) = 0$.
$(-\infty) \div (-\infty)$	Indeterminate	Although signs cancel, both numerator and denominator are unbounded; thus, the limit form remains indeterminate.
∞^∞	∞	An unbounded base raised to an unbounded positive exponent diverges to infinity. Example: $\lim_{x \rightarrow \infty} (x^x) = \infty$.
$\infty \div 1$	∞	Dividing an unbounded quantity by a finite nonzero constant remains unbounded. Example: $\lim_{x \rightarrow \infty} \left(\frac{x}{1}\right) = \infty$.
$1 \div \infty$	0	As the denominator grows without bound, the fraction approaches zero. Example: $\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) = 0$.
∞^1	∞	Any unbounded quantity raised to the power of 1 remains unbounded. Example: $\lim_{x \rightarrow \infty} (x^1) = \infty$.
1^∞	Indeterminate	Although the base is 1, limits of the form (1^∞) can approach values other than 1 depending on the rate of approach.

		Example: $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$, $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{x^2} = \infty$.
∞^{-1}	0	The reciprocal of an unbounded quantity approaches zero. Example: $\lim_{x \rightarrow \infty} x^{-1} = 0$.
$1 \cdot \infty$	∞	Multiplying a finite nonzero constant by an unbounded value remains unbounded. $\lim_{x \rightarrow \infty} 1 \cdot x = \infty$.
∞^0	Indeterminate	The limit form (∞^0) can produce different results depending on how base and exponent vary. Example: $\lim_{x \rightarrow \infty} (x)^{\frac{1}{x}} = 1$, $\lim_{x \rightarrow 0^+} \left(\frac{1}{x}\right)^{\sqrt{x}} = \infty$.
$0 \cdot \infty$	Indeterminate	The product of a vanishing term and an unbounded term depends on their rates of change. Example: $\lim_{x \rightarrow \infty} \left(\frac{x}{x}\right) = 1$, $\lim_{x \rightarrow \infty} \left(\frac{x}{x^2}\right) = 0$.
0^∞	0	Any positive number less than 1 raised to an unbounded power approaches zero. Example: $\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)^x = 0$.
$0 \cdot (-\infty)$	Indeterminate	The same reasoning as $0 \cdot \infty$; the sign depends on context, and the limit is indeterminate.
$0 \div \infty$	0	A vanishing quantity divided by an unbounded one approaches zero. Example: $\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{x} = 0$.
$\infty \div 0$	∞	In the limit form, Dividing infinity by zero approaches infinity. Example: $\lim_{x \rightarrow \infty} \frac{x}{\left(\frac{1}{x}\right)} = \infty$.
$\infty + \pi$	∞	Adding a finite constant to an unbounded value leaves it unbounded. Example: $\lim_{x \rightarrow \infty} (x + \pi) = \infty$.
$\infty \div \pi$	∞	Dividing an unbounded value by a finite positive constant remains unbounded. Example: $\lim_{x \rightarrow \infty} \left(\frac{x}{\pi}\right) = \infty$.
$\pi - \infty$	$-\infty$	Subtracting an unbounded positive value from a finite constant yields an unbounded negative result. Example: $\lim_{x \rightarrow \infty} (\pi - x) = -\infty$.
$\pi \cdot (-\infty)$	$-\infty$	Multiplying a finite positive constant by an unbounded negative quantity gives negative infinity. Example: $\lim_{x \rightarrow \infty} \pi \cdot (-x) = -\infty$.
π^∞	∞	Any constant greater than 1 raised to an unbounded power diverges to infinity. Example: $\lim_{x \rightarrow \infty} (\pi)^x = \infty$.
$\pi^{-\infty}$	0	A finite constant greater than 1 raised to a negative unbounded power approaches zero. Example: $\lim_{x \rightarrow \infty} (\pi)^{-x} = 0$.