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A Qualitative Document Analysis of Quantitative and Covariational Reasoning Opportunities Provided by Calculus Textbooks: The Case of Related Rates Problems

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ABSTRACT

With a focus on related rates problems, the present study reports on quantitative and covariational reasoning opportunities provided by five widely used calculus textbooks in the United States. There are three major results from this study. First, quantitative reasoning opportunities are plentiful, while covariational reasoning opportunities are scarce in all the textbooks, respectively. Second, there is a severe shortage of related rates problems that require more than recalling geometric formulas to mathematize. Third, opportunities promoting the use of diagrams to support students' quantitative reasoning when solving related rates problems are minimal in the practice problems provided in the five textbooks. Overall, the textbooks provide limited opportunities to engage in covariational reasoning when working with related rates problems. Implications for instruction are discussed.

Keywords: Quantitative reasoning, covariational reasoning, related rates problems, derivatives, calculus, document analysis, opportunity to learn

Introduction

A growing number of scholars have called for helping students develop strong quantitative and covariational reasoning abilities, respectively, arguing that this is necessary for students to acquire robust understandings of mathematical concepts/topics that involve making sense of quantities and how these quantities change in relation to each other such as related rates problems in calculus (e.g., Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Moore, 2014; Thompson, 1994, 2011). I remark that related rates problems form an integral part of any first-semester calculus course in the United States (e.g., Engelke, 2007, Engelke-Infante, 2021; Mkhatshwa, 2020a).

A mathematical task is a related rates problem if it involves at least two 'rate' quantities that can be related by an equation, function, or formula (Mkhatshwa, 2020a). There are two types of related rates problems, namely geometric and non-geometric. According to Mkhatshwa (2020a), "a geometric related rates problem is one in which the equation relating the quantities [in the problem] is based on a geometric structure such as the Pythagorean Theorem or the volume of a shape" (p.141). Analogously, a non-geometric related rates problem is one in which the equation relating the quantities in the problem is based on a non-geometric relationship such as some Physics laws (e.g., the ideal gas law) or the economics formula $P = R - C$, where P is profit, R is revenue, and C is cost.

A historical analysis of related rates problems by Austin et al. (2000) reveals that the inclusion of related rates problems in calculus textbooks dates back to at least 1836. Furthermore, these authors found that related rates problems first appeared in the United States in 1851 in a calculus textbook published by Elias Loomis (1811-1889), who was a mathematics professor at Yale University. Students' difficulties with related rates problems, including those directly related to quantitative or covariational reasoning, respectively, are widespread and have continued to be a subject of empirical research investigations for the last 25 years (e.g., Alvine et al., 2007; Azzam et al., 2019; Code et al., 2014; Ellis et al., 2015; Engelke, 2007; Engelke-Infante, 2021; Jeppson, 2019; Kottath, 2021; Martin, 2000; Mkhatshwa & Jones, 2018; Mkhatshwa, 2020a, 2020b; Picollo & Code, 2013; Taylor, 2014; White & Mitchelmore, 1996). In fact, within the last few years alone, several different studies (e.g., Azzam et al., 2019; Engelke-Infante, 2021; Jeppson, 2019; Kottath, 2021; Mirin & Zaskis, 2019; Mkhatshwa, 2020a) have reported on students' difficulties in connection with related rates problems. Evidence from a related line of research suggests the existence of a correlation between learning opportunities provided by mathematics textbooks and difficulties exhibited by students in formal assessments (e.g., Schmidt et al., 2015).

There are generally three different flavors of first-semester calculus offered at the undergraduate level in the United States, namely regular calculus (also known as engineering calculus), life sciences calculus, and business calculus. I note that while related rates problems are a common topic in regular calculus textbooks, they are not covered in most life sciences or business calculus textbooks. The five textbooks considered in this study include three regular calculus textbooks (Stewart et al., 2021; Hughes-Hallett et al., 2021; Rogawski et al., 2019), one life sciences calculus textbook (Greenwell et al., 2015), and one business calculus textbook (Barnett et al., 2019). The research question guiding this study is: What opportunities do calculus textbooks offer students to engage in quantitative reasoning or covariational reasoning when solving related rates problems? I remark that the purpose of this paper is not to make a theoretical contribution, but rather to describe learning opportunities in the context of quantitative reasoning and covariational reasoning provided by calculus textbooks. Additionally, the present study uses the term real-world context broadly to either refer to a relevant and essential context or a camouflage context (e.g., Wijaya et al., 2015). It is worth noting that tasks with the former type of real-world context typically provide more opportunities to engage in quantitative reasoning compared to tasks that have the latter type of real-world context (e.g., Vos, 2020).

Background for the Study

In a recent study (Mkhatshwa, 2022), I reported on quantitative and covariational reasoning opportunities provided by two widely used calculus textbooks in the United States. The focus of the study was on ordinary derivatives and partial derivatives. A key finding of the recent study is that there is a dearth of opportunities to engage in covariational reasoning in connection with ordinary or partial derivatives. Furthermore, the study found that while opportunities to engage in quantitative reasoning are prevalent in one of the textbooks (an applied calculus textbook), there is a short supply of similar opportunities in the other textbook (a traditional calculus textbook).

To ascertain whether the findings of the recent study could be generalized to other topics covered in widely used calculus textbooks in the United States, the present study reports on opportunities to engage in quantitative reasoning and covariational reasoning, in the context of related rates problems, provided by five commonly used calculus textbooks in the United States. I note that two of the five textbooks in the present study were examined in the recent study on ordinary derivatives and partial derivatives. In essence, the present study examined a different topic (related rates problems) compared to the recent study.

Moreover, the present study examined three more textbooks compared to the recent study. As I show in the results section, findings (especially concerning covariational reasoning opportunities) of the present study are very similar to findings of the recent study. The observed similarities in the findings of the two studies lead me to the conclusion that there is generally a paucity of covariational reasoning opportunities in calculus textbooks used in the United States. Arguably, findings of both studies extend beyond the United States, as some of the widely used calculus textbooks in the United States are also used in other countries such as Canada. I therefore argue that calculus textbook authors should strive to include these seemingly lacking opportunities in future editions of their textbooks, in light of the significant role textbooks play in students' learning of mathematics, among other things.

Related Literature

Opportunity to Learn

Although there are slight variations in how the concept of opportunity to learn has been defined in the mathematics education literature, this concept has been used in the same literature for over half a century. For instance, Carrol (1963) defined opportunity to learn as the amount of time devoted to learning about a particular topic, while Husén (1997) defined the same concept as whether or not "students have had the opportunity to study a particular topic or learn how to solve a particular type of problem" (pp. 162-163). According to Floden (2002), Husen's definition of opportunity to learn is commonly used in the mathematics education literature.

This study uses Husén's (1997) definition of opportunity to learn. Specifically, it examined whether or not widely used calculus textbooks in the United States provide opportunities for students to engage in quantitative reasoning or covariational reasoning in the context of working with related rates problems. This examination is particularly important because evidence from a related line of research suggests that student achievement in particular areas/topics of study is tied to the extent to which they have had an opportunity to learn about these areas/topics, such as via classroom instruction or course textbooks (e.g., Cogan & Schmidt, 2015). The significance of textbooks in students' learning of mathematics cannot be overstated. In fact, according to Reys et al. (2004), "the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61), a sentiment that has been echoed by other researchers (e.g., Alajmi, 2012; Kolovou et al., 2009). In this study, the terms "opportunity to learn" and "learning opportunities" are used interchangeably.

An Overview of Mathematics Textbook Research at the K-12 and University Level

Research on learning opportunities provided by mathematics textbooks at the K-12 level (i.e., from Kindergarten to Grade 12) has not only received substantial attention, but also covers a wide range of topics, including cognitive demands of mathematical tasks (e.g., Basyal et al., 2022; Gracin, 2018), deductive reasoning (e.g., Stacey & Vincent, 2009), fractions (e.g., Alajmi, 2012; Charalambous et al., 2010), functions (e.g., Wijaya et al., 2015), problem solving (e.g., Jäder et al., 2020), mathematical reasoning and proof (e.g., Stylianides, 2009; Thompson et al., 2012), probability (e.g., Jones & Tarr, 2007), proportional reasoning (e.g., Dole & Shield, 2008), statistics (e.g., Pickle, 2012), trigonometry (e.g., Wijaya et al., 2015), and students' perceptions regarding the role of textbooks in their learning of mathematics (e.g., Wang & Fan, 2021).

On the contrary, similar research at the undergraduate level has not received considerable attention. The focus of the available studies at the undergraduate level has mainly been on cognitive demands of tasks typically found in mathematics textbooks (e.g., Mesa et al., 2012), learning opportunities related to the concept of the derivative (e.g., Haghjoo et al., 2023, Park, 2016), continuity (e.g., Raman, 2004), optimization problems(e.g., Mkhatshwa & Doerr, 2016; Mkhatshwa, 2023), infinite series (e.g., González-Martín et al., 2011; Heon & Mills, 2023; O'Sullivan et al., 2023), the usage of multiple ways (i.e., algebraically, graphically, numerically, or verbally) to represent mathematical ideas such as functions (e.g., Chang et al., 2016), and limits (e.g., Hong, 2022; Lithner, 2004). On a related note, González-Martín et al. (2018) reported on a case study of how five instructors use a common textbook to prepare for teaching series in calculus. Mesa and Griffiths (2012) described three ways course textbooks mediate the work of college faculty, namely "textbook mediation between instructor and design of instruction" (p. 93), "textbook mediation between instructor and others" (p. 95), and "textbook mediation between instructor and self" (p. 98). According to Mesa and Griffith (2012):

> Reflexive mediation between the textbook and instructors manifests when instructors make mental or physical notes about things that work or do not work, find examples or problems that they need to modify or remove, and identify topics they will not cover or will cover next time they teach (p. 98).

In the context of opportunity to learn, textbook mediation between instructor and self could, for instance, manifest when instructors find or modify examples or problems to supplement essential learning opportunities that are lacking or minimal in the textbooks adopted for their courses. It is worth mentioning that most of the participants in Mesa and Griffiths' (2012) study were calculus instructors. Randahl (2012) reported on how first-year engineering students use mathematics textbooks in their learning of calculus.

The Significance of Textbooks in Mathematics Education

Textbooks play a crucial role in students' learning of mathematics. A recurrent finding from research that has scrutinized the significance of textbooks in the teaching and learning of mathematics at all levels is that nearly all mathematics content covered during classroom instruction is generally dictated by course textbooks (e.g., Begle, 1973; Rezat, 2006; Reys et al., 2004; Robitaille & Travers, 1992; Törnroos, 2005; Wijaya et al., 2015). Indeed, in an attempt to underscore the importance of mathematics textbooks, Begle (1973) asserted that most of what students learn is directed by textbooks rather than teachers. Similar assertions have been echoed by other researchers (e.g., Blazar et al., 2020; Polikoff, 2018; Polikoff et al., 2021).

Students' Difficulties with Engaging in Quantitative Reasoning or Covariational Reasoning when Solving Related Rates Problems

Several studies have reported that related rates problems have a reputation, among students, of being difficult to master (e.g., Alvine et al., 2007; Ellis et al., 2015; Engelke-Infante, 2021). A common finding of research that has examined students' reasoning in the context of working with related rates problems is that students often exhibit difficulties engaging in certain aspects of quantitative reasoning. In particular, a growing number of studies have reported on students who struggled with determining correct units of measure for quantities (e.g., Azzam et al., 2019; Mkhatshwa, 2020a, Kottath, 2021). White and Mitchelmore (1996) reported on students who treated variables representing quantities as symbols that are to be manipulated algebraically and not as quantities that are to be related.

Several studies that have investigated students' thinking about geometric-related rates problems have found that mathematizing (Freudenthal, 1993) this type of problem is problematic for students (e.g., Azzam et al., 2019; Jeppson, 2019; Martin, 2000; Mkhatshwa, 2020a; White & Mitchelmore, 1996). To be specific, mathematizing a related rates problem entails using algebraic symbols to represent the different quantities in the problem, in addition to using an equation/formula to relate the quantities. On a positive note, findings of recent studies on related rates problems suggest that using diagrams to support students' quantitative reasoning is effective when solving related rates problems (e.g., Engelke-Infante, 2021; Mkhatshwa, 2020a).

Evidence from research shows that students exhibit weak covariational reasoning abilities when solving related rates problems (e.g., Carlson et al., 2002; Engelke, 2007). Specifically, this research shows that students seldom engage in the highest levels of covariational reasoning when solving related rates problems. Findings from a related line of research on students' thinking about ordinary derivatives, crucial elements of any related rates problem in calculus, indicate that students' weak covariational reasoning abilities are often evident when they are engaged in solving application problems that involve working with quantities that can be represented using ordinary derivatives (e.g., Jones, 2017; Nagle et al., 2013).

Document Analysis and its Usefulness in Qualitative Research

Document analysis is a useful method in qualitative research (e.g., Merriam & Tisdell, 2016; Morgan, 2022). This study uses the definition of document analysis proposed by Bowen (2009):

> Document analysis is a systematic procedure for reviewing or evaluating documents–both printed and electronic (computer-based and Internet-transmitted) material. Like other analytical methods in qualitative research, document analysis requires that data be examined and interpreted in order to elicit meaning, gain understanding, and develop empirical knowledge [e.g., Corbin & Strauss, 2008; Rapley, 2007]. Documents contain text (words) and images that have been recorded without a researcher's intervention (p. 27).

The documents considered in the present study are five textbooks that are widely used in the teaching of undergraduate calculus in the United States. Factors to consider when selecting documents for analysis include authenticity, credibility, representativeness, and meaning (e.g., Morgan, 2022). According to Morgan (2022), authenticity examines the degree to which a document is genuine, credibility examines the accuracy of a document, representativeness examines the degree to which a document is typical, and meaning examines the degree to which a document's content is significant, clear, or understandable.

I conclude this section by highlighting a few reasons behind my choice of using document analysis in the present study. First, "information and insights derived from documents can be valuable additions to a knowledge base" (Bowen, 2009, p. 30). Second, document analysis does not involve collecting new data. Consequently, the resources (e.g., time and costs) associated with using this methodology are often minimal (Pershing, 2002).

Third, "document analysis can serve as either a stand-alone data-collection procedure or as a precursor to collecting new data using other methodologies" (Pershing, 2002, p. 36). I remark that in the present study, document analysis serves as a stand-alone data-collection procedure. Fourth, the documents are readily available in the public domain (Bowen, 2009). Fifth, document analysis is not affected by obtrusiveness and reactivity i.e. the documents are not affected by the research process (Bowen, 20009). As with any research methodology, document analysis has its own limitations. These include insufficient detail [i.e. documents are not often produced with a research agenda], low retrievability, and biased selectivity [of the documents to be analyzed]. Citing the efficiency and cost-effectiveness of document analysis, Bowen (2009) argued that the benefits of this method far outweigh its limitations.

Theoretical Perspective

Quantitative Reasoning and Covariational Reasoning

Developed nearly three decades ago, the theoretical constructs of quantitative reasoning and covariational reasoning are well known among most mathematics education researchers and practitioners (e.g., Carlson et al., 2002; Smith III & Thompson, 2007; Thompson, 1993, 2011). Consequently, this section provides a synopsis of these theoretical constructs in connection with the present study. The interested reader is referred to my recent study (Mkhatshwa, 2022) for a comprehensive description of the aforementioned theoretical constructs as they relate to the analysis of learning opportunities provided by mathematics textbooks. When measured, quantities have units of measure (e.g., Thompson, 1993). The length of a ladder, the radius of a snowball, and the distance travelled by a car are a few of many examples of quantities referred to in the present study. Quantitative reasoning entails quantification (i.e., determining numeric values for quantities), interpreting quantities, analyzing and determining units of measure for quantities, and analyzing quantities and relationships among quantities based on textual descriptions of problem statements, algebraic equations, graphs/diagrams, or numerical tables of values, respectively, among other things.

Covariational reasoning, on the other hand, deals with analyzing how two or more quantities are changing in relation to each other. Figure 1 provides a description, using the Ladder Problem as an example, of the five levels of covariational reasoning.

Figure 1

Ladder Problem (Reproduced from Carlson et al., 2002, p. 371)

> **From a vertical position against a wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.**

Coordination: At the coordination level of covariational reasoning, also known as Level 1, a recognition that two quantities are changing simultaneously is made. In terms of the related rates problem described in Figure 1, this could mean recognizing that the vertical distance and the horizontal distance are changing simultaneously as the bottom of the ladder is pulled away.

Direction: At the direction level of covariational reasoning, also known as Level 2, attention is given to how two quantities are changing (direction-wise) in relation to each other. This could mean recognizing that as the bottom of the ladder is pulled away, the horizontal distance increases while the vertical distance decreases.

Quantitative Coordination: At the quantitative coordination level of covariational reasoning, also known as Level 3, one coordinates the amount of change of at least one of the two quantities. A qualifying remark at this level could be something like the following: "The vertical distance decreases by 0.5 feet as the horizontal distance increases."

Average Rate: At the average rate of change level of covariational reasoning, also known as Level 4, the focus is on coordinating the average rate of change of one of the quantities with constant changes in the other quantity. A qualifying remark at this level could be a comment like the following: "The vertical distance decreases by 0.75 feet every time the horizontal distance increases by one foot."

Instantaneous Rate: At the instantaneous rate level of covariational reasoning, also known as Level 5, the focus is on coordinating the instantaneous rate of change of one of the quantities with continuous changes in the other quantity. That is, a person reasoning at the instantaneous rate level continuously quantifies how the vertical distance changes with much smaller (less than one foot) changes in the horizontal distance.

The Role of Quantitative Reasoning and Covariational Reasoning in Related Rates Problems

The combination of quantitative and covariational reasoning is crucial in making sense of related rates problems (e.g., Engelke, 2004; Mkhatshwa, 2020a). A multitude of physical or dynamic situations/events can be modelled using related rates problems in different disciplines, including physics, engineering, and economics. In physics, for example, the relationship between the height of a rocket that rises vertically and the angle of a camera placed several yards from the launch pad of the rocket can be modeled using a related rates problem. According to Engelke (2007):

> Solving a related rates problem requires that the student engage in covariational reasoning to understand how the problem works, construct a mental model that allows them to recognize which variables are changing, construct a meaningful relationship between the changing quantities (create an appropriate formula), and reconceptualize the variables in their formula as functions of time. Only then may they use the chain rule to correctly differentiate their formula with respect to time and solve for the desired variable. (p. 29)

Quantitative reasoning plays an important role in the process of solving any related rates problem that has a real-world context. Among other things, the final step when constructing a solution to a related rates problem involves engaging in the process of quantification (i.e., assigning a numerical value to the quantity described by Engelke (2007) as the "desired variable" in the preceding quotation). In fact, some of the previously reported challenges exhibited by students when tasked with solving related rates problems deal directly with quantitative reasoning. Mkhatshwa (2020a) theorized that while covariational reasoning is certainly a key construct when dealing with related rates problems, there may be quantitative ideas, such as the role and use of diagrams to represent relationships between quantities, at play.

As previously noted, there is a relationship between the opportunity to learn about a particular area/topic and students' achievement when assessed in the same area/topic (e.g., Cogan & Schmidt, 2015). Furthermore, both quantitative reasoning and covariational reasoning are essential for students hoping to develop a solid understanding of various calculus ideas, such as the concept of the derivative (e.g., Carlson et al., 2002; Mkhatshwa, 2024). Additionally, covariational reasoning is an essential mode of reasoning for students hoping to make sense of related rates problems in calculus (e.g., Engelke, 2007). According to Jones (2017), Carlson's five levels of covariational reasoning are increasingly sophisticated. Consequently, I posit that students who are able to engage at the highest levels of covariational reasoning demonstrate deeper levels of learning or understanding. Indeed, Carlson et al. (2002) reported on two students (Student A and Student B) who reasoned at the highest levels of covariational reasoning. The students' reasoning at the highest levels of covariational reasoning correlated with high achievement in a related rates task involving a spherically-shaped bottle that was filled with water. In light of the multitude of benefits associated with engaging in quantitative reasoning or covariational reasoning in the study of calculus, it is paramount that calculus textbook authors provide plenty of opportunities (e.g., expository sections, examples, and practice problems) for students to engage in the aforementioned modes of reasoning.

Methods

Analyzed Textbooks

Five textbooks commonly used in the teaching of regular calculus, life sciences calculus, and business calculus in the United States, respectively, were analyzed in this study. See Table 1 for information on the textbooks included for analysis in this study.

Table 1

Analyzed Textbooks

Textbook Name	Author(s)	Sections Analyzed	Textbook Publisher
Calculus: Early Transcendentals (9th ed)	Stewart et al. (2021)	3.9: Related Rates	Cengage Learning
Calculus: Early Transcendentals (4th ed)	Rogawski et al. (2019)	3.10: Related Rates	Macmillan Learning
Single Variable Calculus (8th ed)	Hughes-Hallett et al. (2021)	4.6: Rates and Related Rates	Wiley
Calculus for the Life Sciences (2nd ed)	Greenwell et al. (2015)	6.4: Related Rates	Pearson Education
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	Barnett et al. (2019)	3.7: Related Rates	Pearson Education

Before selecting these textbooks, I consulted with major textbook publishing companies in the United States, including Cengage Learning, Pearson Education, and Wiley, regarding commonly used or ordered calculus textbooks.

Regular calculus in the United States undergraduate mathematics curriculum is generally taken by Science, Technology, Engineering, and Mathematics majors, respectively. Life sciences calculus is typically taken by biology majors, while business calculus is mostly taken by business or economics majors, respectively.

Data Analysis

There are three sources of data for this study, namely (1) expository sections on related rates problems, (2) examples on related rates problems, and (3) practice problems listed at the end of the sections noted in Tables 1 and 2.

Expository sections, examples, and practice problems were analyzed through the theoretical constructs of quantitative reasoning and covariational reasoning, both of which are described in the theoretical perspective section. Additionally, I examined definitions of related rates problems as well as strategies for solving related rates problems as part of my analysis of expository sections. Furthermore, examples or practice problems (hereafter, tasks) were classified as either having real-world contexts or mathematics contexts. In my recent study (Mkhatshwa, 2022), I explained that tasks with the former type of contexts provide opportunities to engage in quantitative reasoning while tasks with the latter type of contexts do not.

Table 2

Counts of Examples, Practice Problems, and Expository Sections

Textbook Name	Section	Expository Sections	Examples	Practice Problems
Calculus: Early Transcendentals (9th ed)	3.9	2	5	53
Calculus: Early Transcendentals (4th ed)	3.10	2	5	45
Single Variable Calculus (8th ed)	4.6	1	4	69
Calculus for the Life Sciences (2nd ed)	6.4	2	6	36
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	3.7	2	4	48
Total		**9**	**24**	**251**

Lastly, evidence from research on students' thinking about related rates problems suggests that students struggle with mathematizing related rates problems (e.g., Azzam et al., 2019; Jeppson, 2019; Martin, 2000; White & Mitchelmore, 1996). Other studies have found that solving non-geometric related problems is particularly challenging for students (e.g., Mkhatshwa, 2020a). Furthermore, findings from research indicate that the use of diagrams could be used to support students' quantitative reasoning when solving related rates problems (e.g., Engelke-Infante, 2021; Mkhatshwa, 2020a). I examined the availability (or lack thereof) of opportunities related to the aforementioned research findings in each textbook.

Illustrations of How Tasks were Coded Through the Lens of Quantitative Reasoning

In this section, I provide examples to illustrate how the tasks (examples and practice problems) were analyzed through the lens of quantitative reasoning.

Practice Problem 2 [Mathematics context] [Non-geometric] (Rogawski et al., 2019, p. 202):

$$\text{If } \frac{dx}{dt} = 2 \text{ and } y = x^3, \text{ what is } \frac{dy}{dt} \text{ when } x = -4, 2, 6?$$

Practice Problem 2 is representative of practice problems I categorized as having a mathematics context, a related rates problem that does not provide quantitative reasoning opportunities such as interpreting quantities, and a non-geometric related rates problem because the equation relating the variables x and y (i.e., $y = x^3$) is not based on a geometric relationship such as the Pythagorean Theorem or the volume of a shape.

Example 3 [Real-world context] [Geometric] (Hughes-Hallett et al., 2021, pp. 254-255)**:** A spherical snowball melts in such a way that the instant at which its radius is $20\ cm$, its radius is decreasing at $3\ cm/min$. At what rate is the volume of the ball of snow changing at that instant?

Example 3 is representative of tasks I categorized as having a real-world context, as a task that requires simple mathematizing as finding the equation that relates the quantities of volume (V)

and radius (r) only requires recalling the volume of a sphere (i.e., $V = \frac{4}{3}\pi r^3$), and as a geometric related rates problem because the equation relating the quantities is based on the volume of a shape (i.e., sphere). Additionally, I categorized Example 3 as a task that provides an opportunity for students to assign a numerical value to the quantity representing the rate at which the volume of the snowball is changing (i.e., decreasing) at the instant when the radius is $20\ cm$. The radius is decreasing at a rate of $3\ cm/min$, and as a task that provides an opportunity for students to determine the units of measure (i.e., cm^3/min) for the aforementioned quantity.

Practice Problem 33 [Real-world context] [Non-geometric] (Barnett et al., 2019, p. 227): Suppose that for a company manufacturing calculators, the cost, revenue, and profit equations are given by

$$C = 90{,}000 + 30x$$
$$R = 300x - \frac{x^2}{30}$$
$$P = R - C$$

where the production output in 1 week is x calculators. If production is increasing at a rate of 500 calculators per week when production output is 6,000 calculators, find the rate of increase (decrease) in: (a) Cost, (b) Revenue, and (c) Profit.

Practice Problem 33 is another example of tasks in the textbooks that I categorized as having a real-world context. I further categorized this task as a non-geometric related rates problem because the equations relating the quantities x, P, R, and C (i.e., $C = 90{,}000 + 30x$, $R = 300x - \frac{x^2}{30}$, and $P = R - C$) are not based on geometric structures. I also categorized Practice Problem 33 as a task that does not require mathematizing as the equations relating the quantities x, P, R, and C are provided. Moreover, I categorized Practice Problem 33 as a task that provides opportunities for students to engage in the process of quantification (i.e., assigning numerical values to the quantities representing the rates at which C, R, and P are changing if production is increasing at a rate of 500 calculators per week when production output is 6,000 calculators). Finally, I categorized this task as providing an opportunity for students to make sense of and to determine the units of measure (i.e., dollars/week) for the aforementioned rate quantities.

Illustrations of How Tasks were Coded Through the Lens of Covariational Reasoning

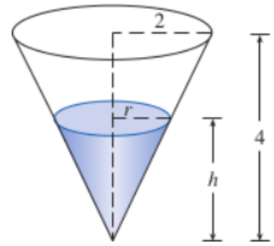
In this section, I provide examples to illustrate how the tasks were analyzed through the lens of covariational reasoning. I begin this section by noting that I categorized tasks that could not be analyzed through the lens of covariational reasoning (e.g., Practice Problem 2, reproduced in the preceding subsection) as tasks that do not provide opportunities to engage in covariational reasoning.

Example 3 [Real-world context] [Geometric] (Stewart et al., 2021, pp. 249-250): A water tank has the shape of an inverted cone with base radius $2\ m$ and height $4\ m$. If water is being pumped into the tank at a rate of $2\ m^3/min$, find the rate at which the water level is rising when the water is $3\ m$ deep.

Solution: We first sketch the cone and label it as in Figure 2. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

Figure 2

Accompanying Diagram-for Example 3



We are given that $\frac{dV}{dt} = 2\ m^3/min$ and we are asked to find $\frac{dh}{dt}$ when $h = 3\ m$. The quantities V and h are related by the equation $V = \frac{1}{3}\pi r^2 h$ but it is very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure 2 to write $\frac{r}{h} = \frac{2}{4}$, from which we get that $r = \frac{h}{2}$. The expression for V becomes $V = \frac{1}{3}\pi h\left(\frac{h}{2}\right)^2 = \frac{\pi}{12}h^3$. Now we can differentiate each side with respect to t: $\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$. Substituting $h = 3\ m$ and $\frac{dV}{dt} = 2m^3/min$, we have $\frac{dh}{dt} = \frac{4}{\pi(3^2)} * 2 = \frac{8}{9\pi}$. The water level is rising at a rate of $\frac{8}{9\pi} \approx 0.28\ m/min$.

Example 3 is representative of tasks I categorized as providing opportunities to engage at the coordination level of covariational reasoning (i.e., Level 1) because the quantities (V, r, and h) are changing simultaneously. The statement [in the solution of the task], "the water level is rising at a rate of $\frac{8}{9\pi} \approx 0.28\ m/min$", provides opportunities to engage at the direction and quantitative coordination levels of covariational reasoning (i.e., Levels 2 and 3). In particular, the remark about the water level rising as time elapses in the aforenoted statement constitutes engaging at the direction level of covariational reasoning. Quantifying the rate at which the water level is rising ($0.28\ m/min$) in the same statement constitutes engaging at the quantitative coordination level of covariational reason. I remark that none of the textbooks were coded at the highest two levels of covariational reasoning (i.e., average rate and instantaneous rate). Specifically, none of the expository sections or tasks included in the five textbooks analyzed in the present study provided opportunities for students to engage at the highest two levels of covariational reasoning.

Additionally, I categorized Example 3 as a task that has a real-world context, as a geometric related rates problem because the equation relating the quantities V, r, and h (i.e., $V = \frac{1}{3}\pi r^2 h$) is based on a geometric structure (i.e., a cone). This is also a task that provides an opportunity to engage in quantification (i.e., finding a numerical value of the rate at which the level of the water is rising), and as a task that requires simple mathematizing as formulating the equation relating the quantities V, r, and h does not require complex reasoning, as in, it can simply be recalled. Furthermore, I categorized Example 3 as a task that provides an opportunity for students to use a diagram (Figure 2) to support their quantitative reasoning when solving the related rates problem in the task. To clarify the coding process, I note that even though there are a few tasks (e.g., Example 3) I analyzed for both quantitative and covariational reasoning opportunities provided in the tasks, for the most part, these two codes (quantitative reasoning and covariational reasoning) are treated as mutually exclusive in the present study. I revisit this issue in the study limitations section of the manuscript.

Results

There are three primary results from this study. First, four of the five textbooks provide concise strategies (i.e., lists of three to seven steps) students could use when solving a related rates problem. The textbook by Hughes-Hallet et al. (2021) is the only textbook that does not provide a list of steps students could follow when solving a related rates problem. Second, all the textbooks provide plenty of opportunities to engage in quantitative reasoning via examples and practice problems on related rates problems. Third, there is a paucity of opportunities to engage in covariational reasoning through expository sections, examples, and practice problems on related rates problems, respectively, provided in all the textbooks. In addition, the few available opportunities are limited to low levels of covariational reasoning, namely coordination, direction, and sometimes quantitative coordination.

Definition of Related Rates Problems and Strategies for Solving these Problems

The definitions of a related rates problem given in the five textbooks are consistent with how a related rates problem is generally understood by the mathematics community in the United States, or how this type of problem is defined in the research literature (e.g., Mkhatshwa, 2020a). Specifically, according to one of the textbooks:

> In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the chain rule to differentiate both sides with respect to time (Stewart et al., 2021, p. 247).

Before giving the aforementioned definition of a related rates problem, Stewart and colleagues (2021) portrayed a picture of a related rates problem by giving the example of pumping air into a balloon. These authors remarked that in this example, it would be easier to measure directly the rate of increase of the volume of the balloon than the rate of increase of the radius of the balloon. These textbook authors went on to propose a seven-step problem-solving strategy (the most comprehensive problem-solving strategy compared to similar problem-solving strategies provided in three other textbooks) that can be used when solving a related rates problem. The following is a reproduction of this strategy (Stewart et al., 2021, p. 249):

> Step 1: Read the problem carefully.
> Step 2: Draw a diagram if possible.
> Step 3: Introduce notation. Assign symbols to all quantities that are functions of time.
> Step 4: Express the given information and the required rate in terms of derivatives.
> Step 5: Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
> Step 6: Use the chain rule [of differentiation] to differentiate both sides of the equation with respect to t [a time variable].
> Step 7: Substitute the given information into the resulting equation and solve for the unknown rate.

I note that the aforementioned steps are similar to steps previously used by students when tasked with solving geometric related rates problems (e.g., Engelke, 2007; Martin, 2000, Mkhatshwa, 2020a). I further note that usage of these steps is well illustrated through five examples in the textbook, one of which was reproduced in the Methods section. I remark that the regular calculus textbook by Rogawski et al. (2019) provides the least comprehensive problem-solving strategy (compared to three other textbooks) that could be used by students when solving a related rates problem. The following is a reproduction of this strategy (Rogawski et al., 2019):

> Step 1: Identify variables and the rates that are related.
> Step 2: Find an equation relating the variables and differentiate it.
> Step 3: Use given information to solve the problem.

Compared to Step 6 in Stewart et al.'s (2021) problem solving strategy, among other things, Step 2 in Rogawski et al.'s (2019) problem solving strategy does not specify the type of differentiation [chain rule] that is to be used after finding the equation that relates the quantities involved in the problem.

Opportunities to Engage in Quantitative Reasoning

Expository Sections. None of the nine expository sections noted in Table 2 provide opportunities to engage in quantitative reasoning. That is, the expository sections in the five textbooks do not provide opportunities to interpret physical quantities, to determine units of measure for physical quantities, or to engage in the process of quantification. I note, however, that all the expository sections came close to providing something we would consider to be opportunities to engage in quantitative reasoning. In the life sciences textbook, for example, Greenwell et al. (2015) posed the following rhetorical question to highlight the importance of related rates problems in the life sciences: When a skier's blood vessels contract because of the cold, how fast is the velocity of the blood changing? These textbook authors went on to make the following remark prior to providing examples on related rates problems:

> It is common for variables to be functions of time; for example, sales of an item may depend on the season of the year, or a population of animals may be increasing at a certain rate several months after being introduced into an area. Time is often present implicitly in a mathematical model, meaning that derivatives with respect to time must be found by the method of implicit differentiation discussed in the previous section (p. 343).

While none of the quantities (e.g., sales of an item, population of animals, the rate of change of the population of animals) needed to be interpreted in the preceding pair of statements, to emphasize the importance of units when making sense of quantities and relationships between quantities, one can argue that Greenwell et al. (2015) could have used, for example, antelopes per year as a unit of measure for the quantity that represents the rate at which the population of animals [e.g., antelopes] is increasing. Similar remarks were made by the authors of the other textbooks considered in this study.

Examples. As can be seen in Table 3, all the examples presented in the related rates section of each of the five textbooks provide ample opportunities to engage in quantitative reasoning (i.e., these examples have real-world contexts). Specifically, the examples generally provide opportunities to interpret physical quantities, to assign units of measure to these quantities, or to engage in the process of quantification. Example 1 is a typical example (in addition to the two Examples 3s that were reproduced in the Methods section) that provides opportunities to engage in the process of quantification, and to make sense of quantities, relationships among quantities, and units of measure for quantities, respectively:

Table 3

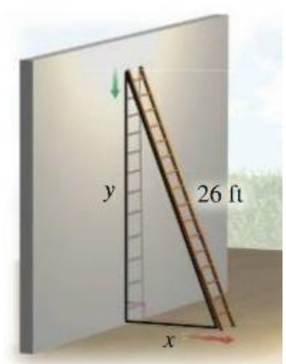
Classification of Examples by Type of Context

Textbook Name	Count of examples with a real-world context	Count of examples with a mathematics context
Calculus: Early Transcendentals (9th ed)	5	0
Calculus: Early Transcendentals (4th ed)	5	0
Single Variable Calculus (8th ed)	4	0
Calculus for the Life Sciences (2nd ed)	6	0
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	4	0
Total	**24**	**0**

Example 1 [Real-world context] [Geometric] (Barnett et al., 2019, p. 222): A 26-foot ladder is placed against a wall as shown in Figure 3.

Figure 3

Diagram that Accompanies Example 1



If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?

This example provides an opportunity to make sense of how several quantities (the vertical distance of the ladder denoted by the variable y in Figure 3, the horizontal distance of the ladder denoted by the variable x in Figure 3, and the rates of change of x and y as the top of the ladder is sliding down the wall) are related. Furthermore, the example provides an opportunity to determine units of measure for the unknown quantity (i.e., the rate at which the bottom of the ladder is moving away from the wall at the instant when the bottom of the ladder is 10 feet away from the wall). It also provides an opportunity to engage in the process of quantification (i.e., determine a numerical value for the quantity that represents the rate at which the bottom of the ladder is moving away from the wall at the instant when the bottom of the ladder is 10 feet away from the wall). Lastly, I interpreted the inclusion of Figure 3 in Example 1 as a means of supporting students' reasoning about relationships among the quantities involved in the example. As can be seen in Table 4, most of the examples in the five textbooks have accompanying diagrams to support students' quantitative reasoning when working through these examples.

Table 4

Count of Examples With or Without Accompanying Diagrams

Textbook Name	Count of examples with accompanying diagrams	Count of examples without accompanying diagrams
Calculus: Early Transcendentals (9th ed)	4	1
Calculus: Early Transcendentals (4th ed)	5	0
Single Variable Calculus (8th ed)	3	1
Calculus for the Life Sciences (2nd ed)	4	2
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	2	2
Total	**18**	**6**

Even though the examples on related rates problems given in the five textbooks are rich in terms of opportunities to engage in quantitative reasoning, as can be seen in Table 5, a majority of the examples are geometric related rates problems.

Table 5

Count of Geometric Versus Non-Geometric Related Rates Examples

Textbook Name	Count of geometric examples	Count of non-geometric examples
Calculus: Early Transcendentals (9th ed)	5	0
Calculus: Early Transcendentals (4th ed)	5	0
Single Variable Calculus (8th ed)	3	1
Calculus for the Life Sciences (2nd ed)	4	2
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	3	1
Total	**20**	**4**

Consequently, mathematizing these examples is straightforward, as it typically involves recalling formulas that relate the quantities involved in these tasks. In Example 1, the equation relating the length of the ladder (26 ft), the quantity representing the vertical distance of the ladder (y), and the quantity representing the horizontal distance of the ladder (x) is given by the Pythagorean Theorem (i.e., $x^2 + y^2 = 26^2$).

Practice Problems. As can be seen in Table 6, a great majority of the practice problems in the five textbooks provide numerous opportunities to engage in quantitative reasoning (i.e., they have real-world contexts). The following is a reproduction of a geometric related rates problem, typical of the five textbooks, that provides opportunities to engage in quantitative reasoning:

Table 6

Classification of Practice Problems by Type of Context

Textbook Name	Count of practice problems with a real-world context	Count of practice problems with a mathematics context
Calculus: Early Transcendentals (9th ed)	51	2
Calculus: Early Transcendentals (4th ed)	43	2
Single Variable Calculus (8th ed)	64	5
Calculus for the Life Sciences (2nd ed)	26	10
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	42	6
Total	**226**	**25**

Practice Problem 13 [Real-world context] [Geometric] (Rogawski et al., 2019, p. 203): At a given moment, a plane passes directly above a radar station at an altitude of $6\ km$.

(a) The plane's speed is $800\ km/h$. How fast is the distance between the plane and the station changing half a minute later?
(b) How fast is the distance between the plane and the station changing when the plane passes directly above the station?

Parts (a) and (b) prompt students to engage in quantification (i.e., to quantify the rate at which the distance between the plane and the station is changing). Students are also expected to determine the units of measure for the specified quantities in parts (a) and (b). Mathematizing this problem (and many other geometric related rates problems found in the five textbooks) is not challenging as it involves using the Pythagorean Theorem. Practice Problem 18, in addition to Practice Problem 33 reproduced in the Methods section, is an example of the few non-geometric related rates problems found in the five textbooks.

Practice Problem 18 [Real-world context] [Non-geometric] (Greenwell et al., 2015, p. 348): The energy cost of horizontal locomotion as a function of the body weight of a marsupial is given by $E = 22.8w^{-0.34}$, where w is the weight (in kg) and E is the energy expenditure (in $kcal/kg/km$). Suppose that the weight of a $10\ kg$ marsupial is increasing at a rate of $0.1 kg/day$. Find the rate at which the energy expenditure is changing with respect to time.

Among other things, Practice Problem 18 provides an opportunity to quantify the unknown quantity (i.e., the rate at which the energy expenditure is changing with respect to time. Furthermore, this practice problem provides an opportunity to make sense of the units of measure for the aforementioned unknown quantity. As with all the other non-geometric related rates problems provided in the five textbooks, students do not have to mathematize this task as the equation [$E = 22.8w^{-0.34}$] relating the quantities E and w is given as part of the statement of the problem. In general, I found no trend in the frequency or amount of available opportunities to work with non-geometric related rates problems presented in the five textbooks. Table 7 displays this information.

Table 7

Count of Geometric Versus Non-Geometric Related Rates Practice Problems

Textbook Name	Count of geometric practice problems	Count of non-geometric practice problems
Calculus: Early Transcendentals (9th ed)	47	6
Calculus: Early Transcendentals (4th ed)	38	7
Single Variable Calculus (8th ed)	35	34
Calculus for the Life Sciences (2nd ed)	12	24
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	25	23
Total	**157**	**94**

Specifically, while the number of non-geometric related rates problems is extraordinarily low in the Stewart et al. (2021) and Rogawski et al. (2019) textbooks, the proportion of geometric and non-geometric related rates problems in the Hughes-Hallett et al. (2021) and Barnett et al. (2019) textbooks is nearly the same. Furthermore, a majority of the problems in the Greenwell et al. (2015) textbook are non-geometric related rates problems. Additionally, opportunities promoting the use of diagrams to make sense of quantities and relationships among quantities while working with related rates problems are extremely low in all five textbooks analyzed in this study. Table 8 displays this information.

Table 8

Count of Practice Problems With or Without Accompanying Diagrams

Textbook Name	Count of practice problems with accompanying diagrams	Count of practice problems without accompanying diagrams
Calculus: Early Transcendentals (9th ed)	10	43
Calculus: Early Transcendentals (4th ed)	14	31
Single Variable Calculus (8th ed)	9	60
Calculus for the Life Sciences (2nd ed)	6	30
Calculus for Business, Economics, Life Sciences, and Social Sciences (14th ed)	2	46
Total	**41**	**210**

Opportunities to Engage in Covariational Reasoning

Expository Sections. Opportunities to engage in covariational reasoning in the nine expository sections (identified in Table 2) on related rates problems in the five textbooks are limited to the lowest two levels of covariational reasoning, namely Level 1 (coordination) and Level 2 (direction). The following is a reproduction, from one of the textbooks, of an exemplary opportunity to engage in covariational reasoning in the expository section of a textbook:

> Union workers are concerned that the rate at which wages are increasing is lagging behind the rate of increase in the company's profits. An automobile dealer wants to predict how much an anticipated increase in interest rates will decrease his rate of sales. An investor is studying the connection between the rate of increase in the Dow Jones average and the rate of increase in the gross domestic product over the past 50 years. In each of these situations, there are two quantities-wages and profits, for example-that are changing with respect to time. We would like to discover the precise relationship between the rates of increase (or decrease) of the two quantities. We begin our discussion of such related rates by considering familiar situations in which the two quantities are distances and the two rates are velocities (Barnett et al., 2019, p. 222).

This remark provides an opportunity to engage in Level 1 of covariational reasoning as it creates an awareness of two quantities (wages and profits) changing in tandem. It also provides an opportunity to engage in Level 2 of covariational reasoning as it speaks of the direction of change (increasing or decreasing) of the aforementioned quantities.

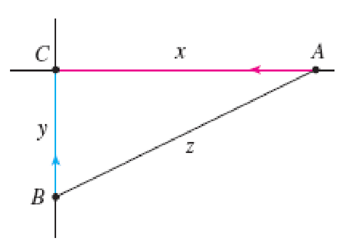
Examples. All the related rates examples from the five textbooks identified in Table 2 provide opportunities to engage in covariational reasoning. However, these opportunities are limited to the lowest levels of covariational reasoning, namely Level 1 (coordination), Level 2 (direction), and Level 3 (quantitative coordination). The following is a reproduction of a typical example from one of the textbooks, in addition to Example 3 that was reproduced in the Methods section, and a discussion of the opportunities to engage in covariational reasoning provided in this example:

Example 4 [Real-world context] [Geometric] (Stewart et al., 2021, p. 250): Car A is traveling west at $50\ mi/h$ and car B is traveling north at $60\ mi/h$. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is $0.3\ mi$ and car B is $0.4\ mi$ from the intersection?

Solution: We draw Figure 4, where C is the intersection of the roads.

Figure 4

Accompanying Diagram-for Example 4



At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles. [It should be noted that although Figure 4 is as a generic diagram that represents the given real-world scenario, in the specific problem given in Example 4, the horizontal distance between A and C is 0.3 miles, and the vertical distance between B and C is 0.4 miles].

We are given that $\frac{dx}{dt} = -50\ mi/h$ and $\frac{dy}{dt} = -60\ mi/h$. (The derivatives are negative because x and y are decreasing). We are asked to find $\frac{dz}{dt}$. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$x^2 + y^2 = z^2$$

Differentiating each side with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dt}{dt}$$
$$<=> \quad \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \qquad z \neq 0$$

When $x = 0.3\ mi$ and $y = 0.4\ mi$, the Pythagorean Theorem gives $z = 0.5\ mi$, so

$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)]$$
$$= -78\ mi/h$$

The cars are approaching each other at a rate of 78 mi/h.

In the preceding solution to Example 4, the statement "at a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars" provides an opportunity to engage in Level 1 (coordination) of covariational reasoning as it provides evidence of the three quantities x, y, and z changing simultaneously with changes in time. In the same solution, the remark that "the derivatives are negative because x and y are decreasing" provides an opportunity to engage in Level 2 (direction) of covariational reasoning. Finally, quantifying the quantities dx/dt and dy/dt through the comment "we are given that $\frac{dx}{dt} = -50\ mi/h$ and $\frac{dy}{dt} = -60\ mi/h$" provides an opportunity to engage in Level 3 (quantitative coordination) of covariational reasoning.

Practice Problems. Practice problems on related rates problems in the five textbooks either do not provide opportunities to engage in covariational reasoning at all such as Practice Problem 2 that was reproduced in in the Methods section, or they provide opportunities to engage at the coordination and direction levels of covariational reasoning (i.e., Levels 1 and 2 of covariational reasoning). Indeed, of the 251 practice problems (see Table 2), 25 practice problems do not provide opportunities to engage in covariational reasoning and the remaining 226 problems provide opportunities to engage at the coordination and direction levels of quantitative reasoning. The following is a reproduction of a representative practice problem from the five textbooks:

Practice Problem 39 [Real-world context] [Geometric] (Hughes-Hallett et al., 2021, p. 260): The radius of a spherical balloon is increasing by $2\ cm/sec$. At what rate is air being blown into the balloon at the moment when the radius is $10\ cm$? Give units in your answer.

This problem provides an opportunity to engage in Level 1 (coordination) of covariational reasoning in that it presents the opportunity to visualize how the quantities of radius and volume are changing in tandem with changes in time. In addition, the question clearly states that the radius is increasing [and while not stated, it can be inferred that the volume is increasing as air is blown into the balloon], thus providing an opportunity to engage in Level 2 (direction) of covariational reasoning. Like Practice Problem 39, most of the practice problems in the five textbooks are focused on calculational knowledge rather covariational reasoning. That is, they tend to emphasize performing calculations over posing questions that promote making sense of how different quantities are changing in relation to each other as time changes.

Discussion and Conclusions

Even though the five textbooks do not have opportunities to engage in quantitative reasoning in their expository sections on related rates problems, the textbooks provide ample opportunities to engage in quantitative reasoning through 24 examples and 226 (out of 251) practice problems on related rates problems, respectively. The prevalence of opportunities to engage in quantitative reasoning in the textbooks is in compliance with growing calls from several researchers and mathematics educators to include such opportunities in undergraduate mathematics education (e.g., Castillo-Garsow, 2012; Moore, 2014; Thompson, 2011). Arguably, the fact that opportunities to engage in quantitative reasoning are plentiful may suggest that some of the previously reported students' difficulties, such as interpreting quantities (e.g., Azzam et al., 2019; Mkhatshwa, 2020a, Kottath, 2021) and making sense of relationships among quantities (e.g., White & Mitchelmore, 1996), with engaging in quantitative reasoning when solving related rates problems may originate from other sources (e.g., classroom instruction), and not necessarily from calculus textbooks.

Findings from previous research on related rates problems indicate that diagrams [pictures of situations] are helpful when solving geometric related rates problems (e.g., Engelke-Infante, 2021; Mkhatshwa, 2020a). In general, it is commendable that all the textbooks considered in this study provide a substantial number of opportunities (via examples) to work with diagrams when solving related rates problems. Specifically, 18 of the 24 examples on related rates problems presented in the five textbooks have accompanying diagrams, thus promoting the use of diagrams when working with related rates problems. On the contrary, opportunities promoting the use of diagrams via practice problems when solving related problems are disproportionately low in all five textbooks. In particular, of the 251 practice problems on related problems found in the five textbooks, only 41 practice problems have accompanying diagrams. I thus recommend that textbook selection committees in mathematics departments consider, among other things, the proportion of examples and practice problems providing opportunities to work with diagrams when adopting calculus textbooks. Similarly, calculus instructors are encouraged to regularly use diagrams (when appropriate) in their teaching of related rates problems in calculus. It would also benefit students if instructors could include explicit prompts on homework assignments or even exams (on related rates), encouraging students to create and use diagrams (when appropriate) to support their quantitative reasoning when solving related rates problems.

Mathematizing a great majority of the geometric-related rates tasks found in the five textbooks is, for the most part, straightforward and often involves using slight variations of the Pythagorean theorem or recalling geometric formulas such as the formula for the volume of a sphere. In addition, nearly all the non-geometric related rates problems found in the five textbooks do not need to be mathematized, as the equations relating the quantities involved in these problems are provided. Of the five textbooks considered in this study, the proportion of non-geometric related rates problems (compared to geometric related rates problems) was extraordinarily low in two of the textbooks, about the same in two other textbooks, and significantly high in one other textbook. A common theme from a growing number of studies on related rates problems is that mathematizing these types of problems is often a challenge for many students in calculus (e.g., Azzam et al., 2019; Jeppson, 2019; Martin, 2000; Mkhatshwa, 2020a; White & Mitchelmore, 1996). To this end, I recommend that calculus textbook authors consider including a fair balance of geometric and non-geometric related rates problems in their textbooks, and most importantly, including related rates problems that require engaging in deep and meaningful aspects of quantitative reasoning that go beyond simply recalling and using geometric formulas when mathematizing these problems. The same consideration applies to calculus instructors in their teaching of related rates problems, or textbook selection committees in mathematics departments, when adopting calculus textbooks for their departments.

Opportunities to engage in covariational reasoning (Carlson et al., 2002) provided in the five textbooks are not only minimal, but also limited to the lowest levels of covariational reasoning, namely coordination, direction, and quantitative coordination. Specifically, I did not find any opportunities to engage in the highest (i.e., more sophisticated) levels of covariational reasoning, namely average rate and instantaneous rate in the expository sections, examples, and practice problems on related rates problems, respectively, included in the five textbooks. Findings from research indicate that students' covariational reasoning abilities are typically limited to the lowest levels of covariational reasoning when solving related rates problems (e.g., Engelke, 2007). Other research has found that students show little or no evidence at all of engaging in covariational reasoning when dealing with derivatives, which are crucial elements of related rates problems (e.g., Carlson et al, 2002; Jones, 2017; Nagle et al., 2013).

In light of the fact that opportunities to engage in covariational reasoning, let alone opportunities to engage in the highest levels of covariational reasoning, are scanty in the five textbooks, I recommend that calculus textbook authors include plenty of opportunities to engage in covariational reasoning when creating expository sections, examples, and practice problems on related rates problems. This is especially important because evidence from research indicates that most student learning is often directed by the textbook rather than the instructor (e.g., Alajmi, 2012; Begle, 1973; Kolovou et al., 2009, Törnroos, 2005; Wijaya et al., 2015). In fact, Reys et al. (2004) posited that the presentation of instructional content during course lectures closely follows the presentation of such content in mathematics textbooks, an argument supported by other scholars (e.g., Blazar et al., 2020; Polikoff et al., 2021). Furthermore, I recommend that textbook selection committees adopt textbooks that provide such opportunities in abundance in light of the crucial role that covariational reasoning plays in students' understanding of calculus topics, including related rates problems. Finally, I recommend that calculus instructors create and use, during classroom instruction, more tasks that could support students in developing strong covariational abilities (i.e., support them in engaging in the highest levels of covariational reasoning). This could include designing tasks that require students not only to create diagrams, but also to make sense of these diagrams to successfully solve related rates problems. Additionally, this might mean calculus instructors will have to design and use related rates problems that have realistic and essential contexts during classroom instruction. This is particularly important because evidence from a recent study on the teaching of related rates problems indicated that related rates problems were not varied and tended to be similar from one calculus textbook to another (Mkhatshwa, 2023).

In conclusion, I note that the five calculus textbooks examined in this study are arguably representative of a great majority of widely used textbooks in the teaching of regular, business, and life sciences calculus, respectively, in the United States. I further note that results from the present study are, to a great extent, consistent with findings from my recent study (Mkhatshwa, 2022) that examined learning opportunities about ordinary and partial derivatives provided by two calculus textbooks. Specifically, both studies have found that calculus textbooks by and large provide enough quantitative reasoning opportunities, and that there is a deficiency of covariational reasoning opportunities, especially opportunities to engage in the highest levels of covariational reasoning, in the same textbooks. Based on the findings of these two studies and a growing number of calls from renowned scholars and educators to include covariational reasoning opportunities in the study of calculus, I appeal to calculus textbook authors to substantially increase covariational reasoning opportunities in their textbooks in virtually every topic (e.g., derivatives, related rates problems, differentials, optimization problems, etc.). This is especially true for opportunities to engage in the highest levels of covariational reasoning, namely average rate and instantaneous rate, which are currently lacking in most widely used calculus textbooks.

Study Limitations

I conclude this paper by highlighting the study limitations. First, the textbooks analyzed in the present study are widely used in the teaching of calculus in the United States. Consequently, findings of the present study may not extend beyond the United States. It might be important for future research to examine similar opportunities to learn provided by other widely used calculus textbooks in other parts of the world. Second, Pershing (2002) remarked that document analysis can be used as a stand-alone data-collection procedure or as a precursor to collecting new data. In this study, document analysis was used as a stand-alone data collection procedure. I, however, posit that there might be added value in using document analysis in conjunction with other methodologies. For instance, it might be helpful to present researchers' findings from conducting a document analysis alongside perspectives of the authors of the documents that were analyzed. In the context of the present study, it would have been beneficial to present the textbooks author's perspectives (obtained via interviews or questionnaires) alongside the results obtained by analyzing the five textbooks examined in the study. Third, the quantitative reasoning and covariational reasoning codes used in the present study are mostly mutually exclusive. In other words, I did not consider tasks that provide opportunities to engage in both quantitative and covariational reasoning in greater detail. This could be a subject for future research. This is especially compelling because these two codes may not necessarily be mutually exclusive. Specifically, geometric related rates problems that are situated in real-world contexts provide opportunities for engaging in the two modes of reasoning, namely quantitative and covariational. Fourth, the data was coded by one researcher. Consequently, inter-rater reliability was not established. Follow-up research will likely involve several researchers to establish inter-rater reliability, among other potential benefits of conducting collaborative research.

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References

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. *Educational Studies in Mathematics*, *79*(2), 239–261.

Alvine, A., Judson, T. W., Schein, M., & Yoshida, T. (2007). What graduate students (and the rest of us) can learn from lesson study. *College Teaching*, *55*(3), 109-113.

Austin, B., Barry, D., & Berman, D. (2000). The lengthening shadow: The story of related rates. *Mathematics Magazine*, *73*(1), 3-12.

Azzam, N. A., Eusebio, M., & Miqdadi, R. (2019). Students' difficulties with related rates problems in calculus. In *2019 Advances in Science and Engineering Technology International Conferences (ASET)* (pp. 1-5).

Barnett, R. A., Ziegler, M. R., Byleen, K. E., & Stocker, C. J. (2019). *Calculus for business, economics, life sciences, and social sciences* (14th ed.). Pearson Education.

Basyal, D., Jones, D. L., & Thapa, M. (2022). Cognitive demand of mathematics tasks in Nepali middle school mathematics textbooks. *International Journal of Science and Mathematics Education*, *24(4),* 1-17.

Begle, E. G. (1973). Some lessons learned by SMSG. *Mathematics Teacher*, *66*(3), 207–214.

Blazar, D., Heller, B., Kane, T. J., Polikoff, M., Staiger, D. O., Carrell, S., ... & Kurlaender, M. (2020). Curriculum reform in the Common Core era: Evaluating elementary math textbooks across six US states. *Journal of Policy Analysis and Management*, *39*(4), 966-1019.

Bowen, G. A. (2009). Document analysis as a qualitative research method. *Qualitative Research Journal*, *9*(2), 27-40.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 55–73). University of Wyoming.

Chang, B. L., Cromley, J. G., & Tran, N. (2016). Coordinating multiple representations in a reform calculus textbook. *International Journal of Science and Mathematics Education*, *14*(8), 1475–1497.

Charalambous, C. Y., Delaney, S., Hsu, H. Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*(2), 117–151.

Code, W., Piccolo, C., Kohler, D., & MacLean, M. (2014). Teaching methods comparison in a large calculus class. *ZDM-International Journal on Mathematics Education*, *46*(4), 589-601.

Cogan, L. S. & Schmidt, W. H. (2015). The concept of opportunity to learn (OTL) in international comparisons of education. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 207-216). Springer.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66–86.

Corbin, J. & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Sage.

Dole, S., & Shield, M. J. (2008). The capacity of two Australian eighth-grade textbooks for promoting proportional reasoning. *Research in Mathematics Education, 10*(1), 19–35.

Ellis, J., Hanson, K., Nuñez, G., & Rasmussen, C. (2015). Beyond plug and chug: An analysis of Calculus I homework. *International Journal of Research in Undergraduate Mathematics Education*, *1*(2), 268-287.

Engelke, N. (2004). Related rates problems: Identifying conceptual barriers. In D. McDougall (Ed.), *26th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.455–462). Toronto, Ontario, Canada.

Engelke, N. (2007). *Students' understanding of related rates problems in calculus*. [Doctoral dissertation, Arizona State University]. ProQuest.

Engelke-Infante, N. (2021). Helping students think like mathematicians: Modeling-related rates with 2 diagrams. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *31*(7), 749-759.

Floden, R. E. (2002). The measurement of opportunity to learn. In A. C. Porter & A. Gamoran (Eds.), *Methodological advances in cross-national surveys of educational achievement* (pp. 231–266). National Academy Press.

Freudenthal, H. (1993). Thoughts in teaching mechanics didactical phenomenology of the concept of force. *Educational Studies in Mathematics, 25*(1&2), 71–87.

González-Martín, A. S., Nardi, E., & Biza, I. (2011). Conceptually-driven and visually-rich tasks in texts and teaching practice: The case of infinite series. *International Journal of Mathematical Education in Science and Technology*, *42*(5), 565–589. https://doi.org/10.1080/0020739X.2011.562310

González-Martín, A. S., Nardi, E., & Biza, I. (2018). From resource to document: Scaffolding content and organising student learning in teachers' documentation work on the teaching of series. *Educational Studies in Mathematics*, *98*, 231-252.

Gracin, D. G. (2018). Requirements in mathematics textbooks: A five-dimensional analysis of textbook exercises and examples. *International Journal of Mathematical Education in Science and Technology*, 49(7), 1003–1024. https://doi.org/10.1080/0020739X.2018.1431849

Greenwell, R. N., Ritchey, N. P., & Lial, M. L. (2015). *Calculus for the life sciences* (2nd ed.). Pearson Education.

Haghjoo, S., Radmehr, F., & Reyhani, E. (2023). Analyzing the written discourse in calculus textbooks over 42 years: The case of primary objects, concrete discursive objects, and a realization tree of the derivative at a point. *Educational Studies in Mathematics*, *112*(1), 73-102.

Heon, N., & Mills, M. (2023). Comparing the textbook with professors' intended and enacted potential intellectual need for infinite series in Calculus II. *Investigations in Mathematics Learning, 15*(3), 169-185.

Hong, D. S. (2022). Examining opportunities to learn limits in widely used calculus textbooks. *International Journal of Science and Mathematics Education*, *21*(4), 1-18.

Hughes-Hallett, D., Gleason, A. M., & McCallum, W. G. (2021). *Single Variable Calculus* (8th ed.). Wiley.

Husén, T. (Ed.). (1997). International study of achievement in mathematics: A comparison of twelve countries (Vol. 2). John Wiley & Sons.

Jäder, J., Lithner, J., & Sidenvall, J. (2020). Mathematical problem solving in textbooks from twelve countries. *International Journal of Mathematical Education in Science and Technology*, *51*(7), 1120-1136.

Jeppson, H. P. (2019). *Developing understanding of the chain rule, implicit differentiation, and related rates: Towards a hypothetical learning trajectory rooted in nested multivariation* [Unpublished master's thesis]. Brigham Young University.

Jones, D. L., & Tarr, J. E. (2007). An examination of the levels of cognitive demand required by probability tasks in middle grades mathematics textbooks. *Statistics Education Research Journal*, *6*(2), 4–27.

Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *The Journal of Mathematical Behavior*, *45*, 95–110.

Kolovou, A., van den Heuvel-Panhuizen, M., & Bakker, A. (2009). Non-routine problem solving tasks in primary school mathematics textbooks—a needle in a haystack. *Mediterranean Journal for Research in Mathematics Education, 8*(2), 31–68.

Kottath, A. (2021). *An investigation of students' application of critical thinking to solving related rates problems* [Unpublished master's thesis]. Oklahoma State University.

Lithner, J. (2004). Mathematical reasoning in calculus textbook exercises. *The Journal of Mathematical Behavior*, *23*(4), 405-427.

Martin, T. (2000). Calculus students' ability to solve geometric related-rates problems. *Mathematics Education Research Journal, 12*(2), 74–91.

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey Bass.

Mesa, V., & Griffiths, B. (2012). Textbook mediation of teaching: An example from tertiary mathematics instructors. *Educational Studies in Mathematics*, *79*, 85-107.

Mesa, V., Suh, H., Blake, T., & Whittemore, T. (2012). Examples in college algebra textbooks: Opportunities for students' learning. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *23*(1), 76–105.

Mirin, & Zazkis. (2019). Making implicit differentiation explicit. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 792–800). Oklahoma City, OK.

Mkhatshwa, T., & Doerr, H. M. (2016). Opportunity to learn solving context-based tasks provided by business calculus textbooks: An exploratory study. In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1124-1132). Pittsburgh, Pennsylvania.

Mkhatshwa, T., & Jones, S. R. (2018). A study of calculus students' solution strategies when solving related rates of change problems. In Weinberg, Rasmussen, Rabin, Wawro, & Brown (Eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 408-415). San Diego, California.

Mkhatshwa, T. P. (2020a). Calculus students' quantitative reasoning in the context of solving related rates of change problems. *Mathematical Thinking and Learning*, *22*(2), 139-161.

Mkhatshwa, T. (2020b). A quantitative reasoning study of student-reported difficulties when solving related rates problems. In Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.), *Proceedings of the 42nd Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 991-995). Mazatlán, Mexico.

Mkhatshwa, T. (2022). Quantitative and covariational reasoning opportunities provided by calculus textbooks: The case of the derivative. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2022.2129497

Mkhatshwa, T. (2023). Calculus instructors' perspectives on effective instructional approaches in the teaching of related rates problems. *Eurasia Journal of Mathematics, Science and Technology Education, 19*(11), 1-15. https://doi.org/10.29333/ejmste/13658

Mkhatshwa, T. (2024). Best practices for teaching the concept of the derivative: Lessons from experienced calculus instructors. *Eurasia Journal of Mathematics, Science and Technology Education, 20*(4), 1-17. https://doi.org/10.29333/ejmste/14380

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102–138.

Morgan, H. (2022). Conducting a qualitative document analysis. *The Qualitative Report*, *27*(1), 64-77.

Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education, 11*(6), 1491–1515.

O'Sullivan, B., Breen, S., & O'Shea, A. (2023). An analysis of Irish mathematics textbook tasks in the context of curriculum change. *Irish Educational Studies*, *43*(4), 1101-1119.

Park, J. (2016). Communicational approach to study textbook discourse on the derivative. *Educational Studies in Mathematics*, *91*(3), 395-421.

Pershing, J. L. (2002). Using document analysis in analyzing and evaluating performance. *Performance improvement*, *41*(1), 36-42.

Pickle, M. C. C. (2012), *Statistical content in middle grades mathematics textbooks*. [Unpublished doctoral dissertation]. University of South Florida.

Polikoff, M. (2018). The challenges of curriculum materials as a reform lever. *Economic Studies at Brookings: Evidence Speaks Reports*, *2*, 58.

Polikoff, M. S., Rabovsky, S. J., Silver, D., & Lazar-Wolfe, R. (2021). The equitable distribution of opportunity to learn in mathematics textbooks. *American Educational Research Association Open*, *7*(1), 1-18.

Raman, M. (2004). Epistemological messages conveyed by three high-school and college mathematics textbooks. *The Journal of Mathematical Behavior*, *23*(4), 389-404.

Randahl, M. (2012). First-year engineering students' use of their mathematics textbook-opportunities and constraints. *Mathematics Education Research Journal*, *24*, 239-256.

Rapley, T. (2007). *Doing conversation, discourse and document analysis.* Sage.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership*, *61*(5), 61–66.

Rezat, S. (2006). The structure of German mathematics textbooks. *ZDM-Mathematics Education, 38*(6), 482–487.

Robitaille, D. F., & Travers, K. J. (1992). International studies of achievement in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 687–709). Macmillan.

Rogawski, J., Adams, C., & Franzosa, R. (2019). *Calculus: Early transcendentals* (4th ed.). Macmillan Learning.

Schmidt, W. H., Burroughs, N. A., Zoido, P., & Houang, R. T. (2015). The role of schooling in perpetuating educational inequality: An international perspective. *Educational Researcher*, *44*(7), 371–386.

Smith III, J. P. J, & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). Erlbaum.

Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics, 72*(3), 271–288.

Stewart, J., Clegg, D., & Watson, S. (2021). *Calculus: Early transcendentals* (9th ed.). Cengage Learning.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11*(4), 258–288.

Taylor, A. V. (2014). *Investigating the difficulties of first year mainstream mathematics students at the University of Western Cape with "related rates" problems* [Unpublished master's thesis]. University of Western Cape.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education*, *43*(3), 253–295.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2), 229–274.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33–57). University of Wyoming.

Törnroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation*, *31*(4), 315–327.

Vos, P. (2020). Task contexts in Dutch mathematics education. In M. Van den Heuvel-Panhuizen (Ed.), *National reflections on The Netherlands didactics of mathematics* (pp. 31–53). Springer.

Wang, Y., & Fan, L. (2021). Investigating students' perceptions concerning textbook use in mathematics: A comparative study of secondary schools between Shanghai and England. *Journal of Curriculum Studies*, *53*(5), 675-691.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*(1), 79–95.

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, *89*(1), 41–65.