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Prospective Teachers' Noticing of Students' Algebraic Thinking: The Case of Pattern Generalization

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ABSTRACT

This study aimed to investigate prospective middle school mathematics teachers' noticing of students' algebraic thinking based on students' correct and incorrect solutions within the context of pattern generalization. Designed as a qualitative case study, three noticing prompts were asked of thirty-two prospective middle school mathematics teachers. Along with it, a semi-structured interview was conducted with eight prospective teachers out of thirty-two prospective teachers. The findings of this study demonstrated that while most prospective teachers could attend to the students' correct and incorrect solutions, they had difficulty interpreting the students' algebraic thinking. The prospective teachers even provided less evidence to interpret the algebraic thinking of the student with the incorrect solution than with the correct solution. Finally, although a vast majority of the prospective teachers could support the algebraic thinking of the student having an incorrect solution, they could not extend the algebraic thinking of the student having a correct solution.

Keywords: correct and incorrect solution; pattern generalization; professional noticing of children's mathematical thinking; students' algebraic thinking; teacher noticing

Introduction

Rather than being a haphazard act, noticing is an intentional act performed consistently within various contexts (Mason, 2011). For more than two decades, many researchers have paid greater attention to how people notice their environment and have approached noticing from a professional point of view (Goodwin, 1994; Stevens & Hall, 1998). For instance, Goodwin (1994) used the term "professional vision" to explain how members of a profession revealed and developed a perceptual framework that allowed them to recognize complicated situations in certain ways. If it is adapted to the context of teaching, professional vision refers to the ability to interconnect theoretical knowledge and practice by noticing the noteworthy events in complex classroom environments (Blömeke et al., 2015; Goodwin, 1994). Professional vision helps teachers notice the classroom environment and, specifically, students' thinking, and it is considered an indispensable skill and a prerequisite for effective teaching (Grossman et al., 2009).

Teacher noticing is an active process rather than a static category of knowledge. It includes the skills of analyzing remarkable events in a classroom setting in which everything simultaneously occurs and the ability to tackle these complex events (Jacobs et al., 2010; Star & Strickland, 2008; van Es & Sherin, 2002, 2021). Teacher noticing also requires teachers to be knowledgeable about the ways students' solutions are not meaningful, as well as to be alert to the correctness of their answers (Holt et al., 2013; Jacobs et al., 2010; van Es, 2011). Thus, teacher noticing, as a skill, is an important dynamic competency required for all teachers, and mathematical teaching could be enriched further by paying more attention to these skills (Franke et al., 2001; Goodwin, 1994; Kaiser et al., 2015).

The significance of teacher noticing skills is not a matter of dispute among scholars, but previous research has explored teacher noticing by confining it to either teachers' correct mathematical thinking (Tyminski et al., 2021) or incorrect mathematical thinking (Copur-Genckturk & Rodrigues, 2021; Girit-Yıldız et al., 2022). Investigating teachers' noticing by focusing on students' both correct and incorrect solutions gives more detailed implications about teachers' expertise. More specifically, as mentioned by Chick et al. (2006), students' correct solutions provided a more significant opportunity to attend to the steps of students' solutions and interpret students' understanding based on the important issues of the related subject. However, previous studies (e.g., Crespo, 2002) indicated that incorrect students' solutions required identifying at what stage students made errors and defining the reasons for making these errors. In order to notice students' incorrect solutions, teachers need to attend to students' errors/difficulties/misconceptions, interpret students' thoughts on the causes of their errors, and decide how to support students' understanding. In other words, teachers need to uncover the reasons for students' incorrect solutions, which enables teachers to make better instructional decisions. In addition, although noticing students' incorrect mathematical understanding is crucial for effective mathematics teaching, not as many studies have been conducted with the intention of noticing students' incorrect reasoning as those focused on identifying students' accurate reasoning (Shaughnessy et al., 2021). Thus, based on the advantages of correct and incorrect responses, the present study focused on prospective teachers' noticing skills of students' correct and incorrect reasoning.

Empirical research has consistently demonstrated that teachers' professional noticing is inherently domain-specific. As a result, it is essential to examine teachers' noticing skills across distinct mathematical domains in order to identify areas requiring development from a subject-specific perspective (Jacobs & Empson, 2016; Ivars et al., 2020; Nickerson et al., 2017). Informed by these findings, the present study investigates the domain-specific nature of professional noticing, with a particular focus on the mathematical context of algebra, for the following reasons. Algebra is one of the key mathematical domains that teachers must attend to and interpret effectively, as it is widely regarded as a gatekeeper in mathematics education due to its foundational role in supporting the development of more advanced mathematical concepts (Blanton & Kaput, 2005; Knuth et al., 2005; Rakes et al., 2010). Additionally, the algebra domain offers a particularly productive context to investigate teachers' noticing based on students' correct and incorrect answers because algebra, by its nature, provides both correct and incorrect examples, which in turn leads to better learning performances for students (Curry, 2004; Jurdak & El Mouhayar, 2014; Lannin et al., 2006). Therefore, the current study aimed to explore how prospective middle school mathematics teachers notice students' algebraic thinking based on their correct and incorrect solutions.

Theoretical Framework

A great deal of research on teacher noticing was predominantly conducted by drawing on two main theoretical frameworks: "Learning to Notice" and "Professional Noticing of Children's

Mathematical Thinking." Within the Learning to Notice framework, van Es (2011) focused on two dimensions with four levels of teacher noticing: "what teachers notice" and "how teachers notice." The former is related to teachers' observation of students' understanding as a group, classroom environment, and teachers' pedagogy, whereas the latter is related to how teachers analyze and evaluate what they observe (van Es, 2011). van Es described the levels of both dimensions from general to specific, i.e., baseline, mixed, focused, and extended levels. Later, van Es and Sherin (2021) expanded their framework by taking into consideration that noticing was an active process and took place in a context. The revised framework consists of students' understanding and interpretation of solution strategies as well as the shaping of the new dimension.

On the other hand, Jacobs et al. (2010) focused on the fourth level of van Es's framework, which was defined as understanding particular students' thinking and teachers' in-the-moment decisions while responding to students based on their mathematical thinking. In that respect, they put forth "Professional Noticing of Children's Mathematical Thinking" by centering on students' thinking. Since the researchers of the present study aimed to focus on prospective teachers' noticing from the point of a particular student's thinking through students' written solutions rather than the whole classroom setting, the study was grounded on professional noticing of students' mathematical thinking. Written works/solutions serve as an authentic activity for interpreting students' thinking and responding to students based on their thinking for mathematics teaching (Grosman et al., 2009; Jacobs & Philipp, 2004). Thus, given that prospective teachers are teachers in the future, examining their noticing skills through students' written solutions is essential. By building the current study on this framework, the researchers aimed to fill the gap in the relevant literature as to the extent to which prospective teachers noticed students' both correct and incorrect written solutions in the context of a particular mathematical domain, that was, algebra. Similar to Jacobs et al.'s (2010) study, how prospective teachers capture the mathematically noteworthy details in students' written solutions, how they presented evidence regarding their thoughts when evaluating students' written solutions, and how they used this interpretation when responding to students were emphasized.

Professional Noticing of Children's Mathematical Thinking

Professional noticing of children's mathematical thinking, which the current study was preoccupied with, focuses on how and to what extent teachers notice children's mathematical thinking rather than what teachers notice (Jacobs et al., 2010). Professional noticing of children's mathematical thinking consists of three important components: "(1) attending to children's strategies, (2) interpreting children's understanding, and (3) deciding how to respond based on children's understandings" (Jacobs et al., 2010, p. 169). Attending to children's strategies is related to teachers' identification of remarkable mathematical essence in children's strategies (Jacobs et al., 2010). Jacobs et al. (2010) classified teachers' attending skills as proof of whether they attended to children's strategies.

On the other hand, interpreting children's understanding is associated with teachers' analysis and interpretation of children's mathematical understanding based on their strategies (Jacobs et al., 2010). Finally, deciding how to respond, based on children's understanding, is tied to teachers' decisions to respond to children and teachers' reasoning for their decisions (Jacobs et al., 2010). Within this framework, Jacobs et al. (2010) categorized teachers' skills of interpreting and deciding how to respond into three areas: robust evidence, limited evidence, and lack of evidence. Therefore, it would not be wrong to suggest that Jacobs and his colleagues were interested in teachers' noticing each child's mathematical understanding and teachers' in-the-moment decisions to respond to the child rather than focusing on the whole group's mathematical thinking, teacher's pedagogy, or classroom environment (Jacobs et al., 2010; LaRochelle, 2018).

Algebraic Thinking and Pattern Generalization

Algebra is considered a foundation for conceptualizing many advanced mathematical concepts, and it comprises abilities like how variables relate to one another, generalizing that relationship, and using that generalization to formulate a rule using an algebraic expression (Kaput, 1999). The National Council of Teachers of Mathematics (NCTM, 2000) established the objectives that students must meet in order to master algebra under this definition. These objectives included comprehending patterns, relationships, and functions as well as applying algebraic symbols to analyze mathematical situations and structures. These objectives also involved studying change in various contexts and using mathematical models to describe and interpret quantitative relationships. In order to achieve these goals, students need to have and develop their algebraic thinking, which is defined as interpreting symbols and algebraic operations as arithmetic (Kieran & Chalouh, 1993) and being able to make sense of unknown quantities as known quantities with different representations (Swafford & Langrall, 2000).

One of the practical methods of mathematical reasoning that aids students in the transition from arithmetic to algebra is designated as algebraic thinking (Radford, 2008). In other words, students should first be introduced to algebra through an operational perspective before advancing to a structural understanding (Carraher & Schliemann, 2007; Sfard, 1995). This transition typically occurs via a pattern generalization process, which consists of three distinct phases (Radford, 2008; Stacey, 1989):

1. Near Term Generalization: Identifying a recurring process using a step-by-step approach, including drawing and counting.
2. Far-Term Generalization: Extending the generalization to address problems that exceed the limitations of the step-by-step method, such as determining the number of elements in the 80th figure of a pattern.
3. Rule Formulation: Developing a formal rule or formula to describe and define the sequence.

Through the process of pattern generalization, students can express relations that are expressed arithmetically with letters, which results in algebraic thinking. Pattern generalization, expressing the relationship between variables algebraically, is a challenging process for students due to the necessity for a step-by-step solution (Jurdak & Mouhayar, 2014). However, if teachers can comprehend how students construct symbols in their minds and generalize the pattern algebraically, they can create a more effective learning environment for algebra. Therefore, teachers' professional noticing of students' algebraic thinking is crucial for teachers to enhance students' algebraic thinking and teach algebra more effectively (Radford, 2008).

The Rationale of the Study

Relevant literature demonstrates that a great deal of research has been conducted to investigate how teachers notice students' mathematical thinking within specific mathematical contexts (Kılıç & Doğan, 2021; Lee, 2019; Sánchez-Matamoros et al., 2019; Taylan, 2017). In prior studies conducted within the context of algebra, researchers explored teachers' professional noticing of children's algebraic thinking through either video club meetings or student work (LaRochelle et al., 2019; Walkoe, 2013; Zapatera & Callejo, 2013). Even though the current study acknowledged such studies and aimed to attain a similar objective, exploring prospective teachers' noticing skills by utilizing students' both correct and incorrect solutions made the study significant and contributed to relevant literature.

Correct and incorrect solutions have diverse attributes, so teachers must highlight different aspects of students' correct and incorrect solutions to notice their mathematical thinking. First, to notice students' correct solutions, prospective teachers have to pay attention to different ways to solve a problem and analyze how students think. On the other hand, to notice students' incorrect solutions, prospective teachers need to attend to students' conceptual and procedural mistakes/misconceptions and understand the reasons why these students have such difficulties. Second, due to the nature of the pattern generalization, students must solve the problem step-by-step in order to formulate a general rule (Jurdak & El Mouhayar, 2014; Lannin et al., 2006). Thus, it can clearly be observed how students arrive at the correct solution through a step-by-step process.

However, it is challenging to determine at which stages of pattern generalization students make mistakes and/or have misconceptions that lead them to incorrect solutions. Moreover, students' correct and incorrect solutions have critical roles in examining whether teachers can extend/support the mathematical thinking of students with correct and incorrect solutions (Jacobs et al., 2010). Finally, researchers who have investigated students' thinking through video club meetings or student work focus on either only students' correct mathematical thinking (Tyminski et al., 2021) or incorrect mathematical thinking (Copur-Gencturk & Rodrigues, 2021; Girit-Yıldız et al., 2022). Therefore, it is crucial to examine prospective teachers' noticing of students' algebraic thinking using both correct and incorrect solutions to portray the whole picture of teachers' expertise in attending, interpreting, and deciding how to respond. Furthermore, Jacobs and Ambrose (2008) and Milewski and Strickland (2016) examined teachers' moves to improve students' thinking using two different categorizations: correct and incorrect answers. This categorization also proves that investigating teachers' noticing of students' both correct and incorrect solutions is significant.

Moreover, the categorization of Jacobs et al.'s framework did not cover all the data gathered from the prospective teachers. For this reason, it was necessary to extend Jacobs et al.'s framework to enable a detailed analysis of all skills. The first component of professional teacher noticing -attending to students' solutions- includes two categories: evidence of attending and lack of evidence of attending (Jacobs et al., 2010). However, in this study, some prospective teachers' responses could not be categorized under the evidence of attending or lack of evidence. Thus, to categorize all prospective teachers' responses, two more categories, namely emerging evidence and limited evidence of attending to students' solutions, were added based on the common characteristics of responses. The second component of professional teacher noticing -interpreting students' algebraic thinking- is analyzed under three categories: robust evidence, limited evidence, and lack of evidence (Jacobs et al., 2010). However, because some participants' responses did not match the characteristics of robust evidence or limited evidence, there was a need to add one more category, emerging evidence, between robust and limited evidence.

Furthermore, the third component of professional teacher noticing -deciding how to respond- includes three categories: robust evidence, limited evidence, and lack of evidence (Jacobs et al., 2010). However, this study revealed that prospective teachers either asked questions to develop and extend students' existing understanding or they posed structurally similar questions that were repetitive in nature and failed to connect with or extend the student's current thinking. Therefore, the participants' skill of deciding how to respond was coded under three categories: extending/supporting students' algebraic thinking, *reinforcing procedural understanding*, and providing a general response. Finally, this categorization prepared for student's correct and incorrect solutions separately, which made the present study necessary. Detailed information about the categories used in this study was given in Table 1-2-3 in the findings section.

Lastly, using data from a natural classroom environment instead of taking student solutions from the literature could contribute to the scholarship about students' strategies in pattern generalization tasks. In order to put students' solutions to the questionnaire for teachers, a problem about pattern generalization was asked of 115 6th-grade students. Among their solutions, two solutions (one correct and one incorrect), including noteworthy mathematical details, were used to collect data from the prospective teachers. Therefore, in this research study, how prospective teachers attended to real student solutions obtained from math classes, interpreted students' algebraic thinking, and the nature of their decisions to respond to students were examined. Thus, the following research questions guided the research study:

1. How do prospective middle school mathematics teachers attend to students' correct and incorrect solutions of pattern generalization?
2. How do prospective middle school mathematics teachers interpret students' algebraic thinking based on students' correct and incorrect solutions within the context of pattern generalization?
3. What is the nature of the decisions that prospective middle school mathematics teachers make to respond based on students' correct and incorrect algebraic thinking within the context of pattern generalization?

Methods

Research Design

This study aimed to offer a deeper systematic examination of prospective middle school mathematics teachers' noticing of students' correct and incorrect solutions, so a qualitative case study was decided to be an appropriate research design to undertake such a study (Creswell, 2007; Merriam, 1998). A case study focuses on the process, context, and discovery instead of outcomes and specific variables. In addition, it enables an in-depth understanding of an issue through the opportunity of detailed data collection and analysis (Creswell, 2007; Merriam, 1998). In this sense, the case of this study was a group of prospective middle school mathematics teachers all studying their last year at the university at the same time, and the units of the analysis were the teachers' skills of attending to students' solutions, interpreting students' mathematical understanding, and deciding how to respond. Since there was a single case and three units of analysis (Yin, 2009) in the present study, the model of the single-case embedded design was preferred.

Questionnaires and semi-structured interviews are essential data collection tools to construct case studies appropriately (Merriam, 1998). "Open-ended questions will result in more detailed and useful data than questions that can be answered with a yes or no" (Moore et al., 2012, p.256). For this reason, open-ended questions in questionnaires and interviews facilitate in-depth understanding and detailed insights into participants' thoughts, experiences, and perspectives (Moore et al., 2012; Savin-Baden & Major, 2013). Thus, utilizing questionnaires and semi-structured interviews through open-ended questions is significant for obtaining rich data, analyzing the data meaningfully, and understanding the case comprehensively (Merriam, 1998).

Consequently, in this study, to conduct a case study effectively and investigate the topic deeply, the data for this study were collected from thirty-two prospective teachers through a questionnaire that included open-ended questions about various students' solutions. Thus, the aim was to answer the research questions with a wide range of data and to understand prospective teachers' attending, interpreting, and responding to diversity. In the second stage, semi-structured interviews were conducted with eight of them to obtain more in-depth information. Participants were selected for the interview based on the criteria of volunteering to participate and allocating an appropriate time for the interview. In addition, the participants' capacity to express their thoughts was also taken into consideration. In this context, their instructors' feedback was utilized to conclude that the participants expressed their thoughts more clearly and in detail. As a result, interviews were conducted with eight participants who met the mentioned criteria. Thus, after more in-depth responses in the interview from selected participants, it was ensured that the data provided a broad perspective and in-depth analysis that better supported the study's findings. Therefore, as an essential requirement of the case study, we had the opportunity to hear the prospective teachers' responses on how to notice students' solutions with complete clarity through the questionnaire, consisting of open-ended questions, and examine in-depth and verify their answers in the questionnaire through semi-structured interviews. In conclusion, the findings from both data collection methods were complementary and allowed us to comprehensively understand the prospective teachers' noticing of students' algebraic thinking.

Context and Participants

The current study concentrated on a fourth-year middle school mathematics teacher education (undergraduate) program at a public university in Ankara/Turkey. The program aims to train prospective teachers to gain competencies in improving students' problem-solving skills through critical thinking and teaching mathematics effectively by incorporating technology. The prospective teachers attend elementary mathematics education courses (e.g., Methods of Teaching Mathematics I-II and Nature of Mathematical Knowledge for Teaching), content courses (e.g., Calculus, Statistics, and Physics), and education sciences courses (e.g., Educational Psychology and Classroom Management). The prospective teachers complete most of the content courses in the first two years of this program, while taking education sciences courses and elementary mathematics education courses in the following years.

Participants were selected from one of the top universities in Türkiye through a purposeful sampling method to obtain rich data. Thirty-two prospective middle school mathematics teachers studying in their senior year participated in this study. In addition to many content and educational science courses, most participants completed Methods of Teaching Mathematics I-II and School Experience courses. In the Methods of Teaching Mathematics I-II courses, prospective teachers acquire knowledge on instructing students on mathematical topics and effective teaching methods. Moreover, they reflect on students' potential misconceptions while learning mathematics and discuss appropriate ways to address students' misconceptions.

Thus, the participants were familiar with students' possible conceptual confusion in algebra and algebraic thinking, as well as effective instructional strategies for teaching algebra to middle school students. In the School Experience course, on the other hand, they are given the chance to observe the actual classroom environment and lectures offered by mentor teachers and other prospective teachers. Moreover, as prospective teachers are responsible for giving lectures to an actual classroom within the scope of the School Experience course, the participants had the opportunity to practice teaching mathematics to students and receive feedback from their university instructor and mentor teacher in the middle school.

The content of Methods of Teaching Mathematics I-II and School Experience courses were not primarily designed to develop the prospective teachers' noticing skills. Instead, these courses aimed to enhance their knowledge about teaching mathematics topics effectively, centering on students' understanding and using this knowledge to teach any topic in the actual classroom as a part of their School Experience course. Since prospective teachers who took these courses can provide more extensive data on their noticing skills, participants were chosen from the prospective teachers who completed Methods of Teaching Mathematics I-II and School Experience courses.

Data Collection

This study's data was obtained through three different data collection tools: a questionnaire for 6th-grade students, a questionnaire for prospective teachers, and a semi-structured interview.

Questionnaire for 6th-Grade Students

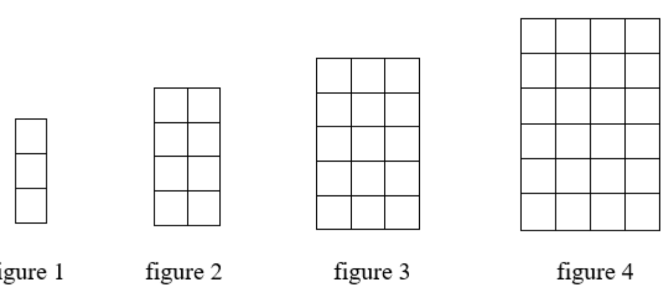
To examine the prospective teachers' professional noticing of students' algebraic thinking based on student works in detail, different students' solutions were needed. Thus, a questionnaire involving three open-ended questions regarding pattern generalization was applied to twenty 6^{th}-grade students to obtain these alternatives. As the objective, "Students should be able to express the rule of arithmetic sequences by using letters and find the desired term of sequences expressed in letters." The standard 6.2.1.1 (MoNE, 2013) is in the 6th-grade mathematics curriculum, so it was determined that the questionnaire be asked of 6th-grade students. Additionally, to comprehensively investigate teachers' noticing of students' algebraic thinking, it was essential to gather solutions, including correct and incorrect steps, mathematically noteworthy details, and exhibiting variability. Thus, the questionnaire for 6th-grade students was administered to students with varying alternative solutions and algebraic thought processes, ensuring the collection of solutions that met these specific criteria. Consequently, this approach enabled alternative 6th-grade student solutions to have a deep evaluation of teachers' noticing of students' solutions.

Since professional noticing of children's mathematical thinking comprised three skills and the study's primary aim was to investigate the prospective middle school mathematics teachers' professional noticing of students' algebraic thinking under three dimensions in depth, the researchers deliberately zoomed in on one of the three questions. One of the criteria for selecting the question was whether it was solved both correctly and incorrectly by the 6th-grade students. Although three questions were related to pattern generalization, this study focused on near and far generalizations since making near and far generalizations was considered a springboard for writing the rule of a pattern (Radford, 2008). For this reason, Question 1 (see Figure 1) was selected to examine prospective teachers' professional noticing in detail.

Figure 1

Question 1 (Radford, 2000)

The first four steps are given in the picture below. The rules of the 5^{th} step and other next steps in the pattern are the same as the rules in the first four steps. According to these steps, find the number of squares in the 25^{th} step. While finding the result, please draw a table and write the algebraic expression.



Questionnaire for Prospective Teachers

After applying the questionnaire to the students, one incorrect (Figure 2) and one correct (Figure 3) student solution was selected in accordance with the aim of this study. The reasons for selecting these solutions were that correct and incorrect solutions had different mathematical nuances to assess the prospective teachers' noticing skills (Jacobs et al., 2010), and these solutions reflected students' alternative thinking, which were worthy of noticing. The students' solutions are represented in the following figures:

Figure 2

Student A's Solution (Incorrect)

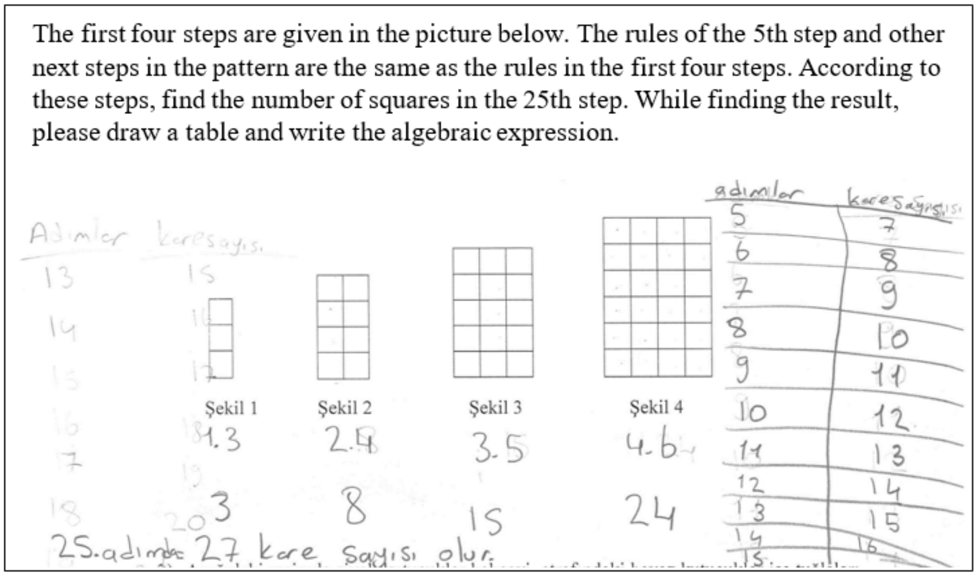
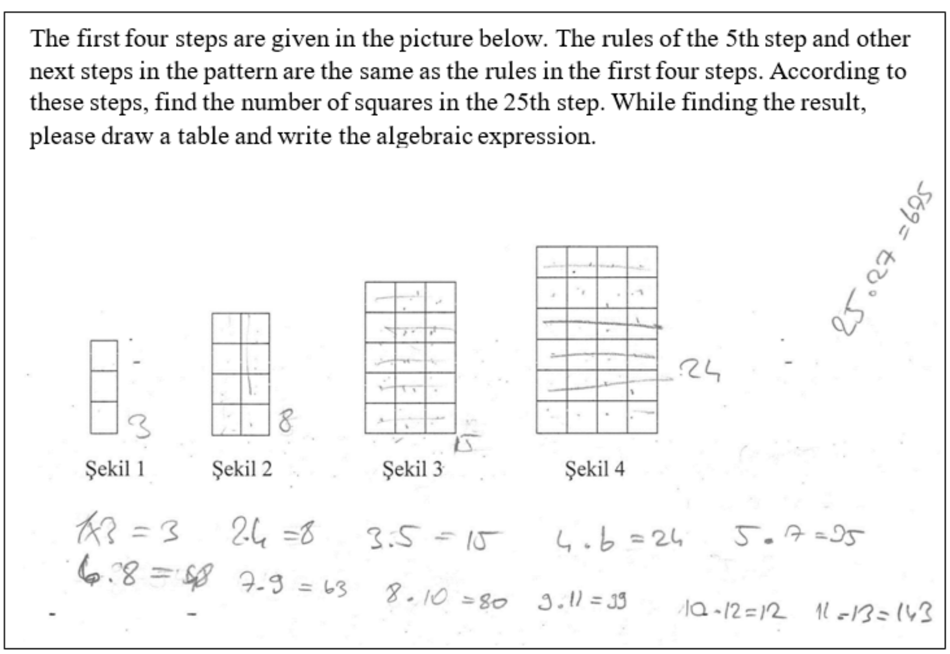


Figure 3

Student B's Solution (Correct)



The questionnaire involving three prompts, initially proposed by Jacobs et al. (2010), was implemented on thirty-two prospective teachers to investigate their skills of attending, interpreting, and deciding how to respond in relation to two students' solutions:

(1) "Please explain in detail what you think each child did in response to this problem.
(2) Please explain what you learned about these children's understanding.
(3) Pretend that you are the teacher of these children. What problem or problems might you pose next?" (Jacobs et al., 2010, 178-179).

Semi-structured interview

Semi-structured interviews offered a flexible structure, allowing participants to freely express their ideas while enabling in-depth exploration of specific issues of the research (Merriam, 1998). For this reason, this data collection tool was preferred to allow participants to express their responses in more detail and to seek answers to the research questions from a broader perspective after implementing the questionnaire. Eight participants were carefully selected for these interviews after implementing the questionnaire. Eight participants who volunteered, had time to participate in the interview, and could express their thoughts comfortably were selected, and an in-depth examination was made through interviews.

Necessary permissions were taken from the Human Subjects and Ethics Committee at the institution where the questionnaire and interviews were applied. Prior to collecting the data, information regarding the study was explained to each participant, and a consent form was taken from volunteer participants. Afterward, the researchers ensured that their personal details, responses, and video recordings would be kept confidential. Finally, a comfortable classroom environment was provided for the participants to answer the questionnaire and conduct interviews.

Data Analysis

In this study, a questionnaire was administered to thirty-two prospective teachers, and interviews were conducted with only eight of them. In data analysis, the responses provided in the questionnaire were primarily utilized. On the other hand, the responses of the prospective teachers, who both answered the questionnaire and were interviewed, were analyzed by considering the two data collection sources. Given that the responses of the prospective teachers who both answered the questionnaire and were interviewed were found to be parallel, their data was evaluated overall by considering the responses provided by both data collection tools. Therefore, some prospective teachers' responses given as examples in the finding sections were excerpted from a questionnaire, and some of them were taken from the interviews.

Data were analyzed according to the dimensions of the Professional Noticing of Children's Mathematical Thinking framework developed by Jacobs et al. (2010). As categories in this framework were insufficient to cover all the data of the present study, new categories were added, and some categories were split into subcategories based on the similarities and differences of the participants' responses. Two mathematics educators specializing in teacher noticing coded the data as co-coders to ensure inter-reliability. The co-coders' and the researchers' codes were compared to identify similarities and differences. Interrater reliability was determined at approximately 93% using the formula outlined by Miles and Huberman (1994). After another discussion about the discrepancies, the required adjustments were made, and ultimately agreement was reached. Consequently, two more categories – emerging evidence and limited evidence – were added to the dimension of attending, and one more category – emerging evidence – was added to the dimension of interpreting. In this way, the two dimensions of teacher noticing, attending and interpreting, could be investigated in greater detail by classifying them under four categories*: robust evidence, emerging evidence, limited evidence,* and *lack of evidence*. On the other hand, in this study, when the relevant data was analyzed, it was observed that prospective teachers either asked questions to develop and extend students' existing understanding or they asked structurally similar questions that were repetitive in nature that were unrelated to the student's current understanding. Prospective teachers' responses forced the researchers to categorize the third component of teacher noticing differently than Jacobs et al. (2010). Therefore, in order to reveal the characteristics of prospective teachers' responses better, the ability to decide how to respond was categorized under three sub-headings: *extending/supporting students' algebraic thinking, reinforcing procedural understanding,* and *providing a general response*. The findings section of the study contained more comprehensive information on these categories and the findings associated with them.

Findings

The findings of this study were presented under three dimensions: attending to students' solutions, interpreting students' algebraic thinking, and deciding how to respond based on the students' algebraic thinking.

Attending to Students' Solutions

This section presents the findings to answer the first research question related to prospective middle school mathematics teachers' attending to students correct and incorrect solutions of pattern generalization. In Jacobs et al.'s (2010) framework, the first component of professional teacher noticing, attending to students' solutions, comprises two categories: evidence of attending and lack of evidence of attending. However, this limited binary structure needed to be revised to adequately reflect the subtle differences in prospective teachers' attending in the current study, which hindered the in-depth analysis of the findings. Consequently, a four-category rating system was created, incorporating two new categories considering the typical characteristics of participants' responses. These additional categories allowed for a more detailed description of the different levels of attending to students' solutions by explaining the transitions in more detail. The properties of the categories related to the dimension of attending and the frequency of prospective teachers' responses are illustrated in Table 1.

Table 1

Details of the Dimension of Attending to Students' Solutions and the Frequency of Each Category

	Attending to Students' Solutions	Frequency
Student A Solution (incorrect)	*Robust Evidence of Attention to Students' Solution* Correctly identifying both how the student finds the number of squares in the first four steps and the student's mistake in creating the table.	17 (53.13%)
	Emerging Evidence of Attention to Students' Solution Correctly identifying how the student finds the number of squares in the first four steps, but the student's mistake in creating the table is missing.	11 (34.38%)
	Limited Evidence of Attention to Students' Solution Correctly identifying the student's mistake, but the explanation of the solution is not in detail.	3 (9.37%)
	Lack of Evidence of Attention to Students' Solution Describing the solution as correct.	1 (3.13%)

	Attending to Students' Solutions	Frequency
Student B Solution (correct)	*Robust Evidence of Attention to Students' Solution* Correctly identifying how the student finds the number of squares in the first four steps and finds the 25th figure.	16 (50%)
	Emerging Evidence of Attention to Students' Solution Correctly identifying how the student finds the number of squares in the first four steps but how the student concludes the solution is missing.	6 (18.75%)
	Limited Evidence of Attention to Students' Solution Correctly identifying student's result but the explanation of the solution is not in detail.	7 (21.88%)
	Lack of Evidence of Attention to Students' Solution Describing the solution as incorrect.	3 (9.37%)

Robust Evidence

More than half of the prospective teachers provided robust evidence to attend to student A's solution (53.13%) and student B's solution (50%), as they described all important mathematical details of the students' solutions. For instance, PT 2's explanation of student A's solution, taken from the questionnaire, was as follows:

> Wrong. In each figure, the student multiplied the number of rows and the number of squares in each row in that figure. When s/he was solving the 5th step, s/he wrote the number of rows in the 5th step instead of writing the total squares in that step. In other words, s/he started the solution with correct reasoning, but when s/he transferred the information to the table, s/he wrongly continued it. S/he continued with the 25th step and said that there were 27 squares in the 25th step since the difference between the number of steps and the number of rows in that step was 2.

PT 2 identified student A's solution as calculating the number of squares in each step by multiplying the number of columns and the number of rows. PT 2 also recognized student A's mistake in creating a table, which led to the incorrect result.

Emerging Evidence

While eleven prospective teachers (34.38%) attended to student A's solution by providing emerging evidence, six of them (18.75%) provided emerging evidence to attend to student B's solution, giving descriptions consisting of mathematically important details but not capturing all the details of the student solutions. PT 9's attending to student A's solution in the questionnaire was presented in the following way:

> The student realized that the number of rows in each step was two more than the number of steps, and s/he concluded his/her solution by stating that there were 27 squares in the 25th step. Student A's solution is wrong because 27 is not the number of squares in the 25th step; actually, it is the number of rows.

PT 9 *described how student A built the relationship* between the number of steps and the number of columns, stating that the student added two to 25. However, PT 9 did *not pay attention to student A's mistake* in transforming knowledge to the table.

Limited Evidence

Three prospective teachers (9.37%) attended to student A's solution, while seven of them (21.88%) attended to student B's solution by providing limited evidence because their explanation included a general description of the students' solutions and did not provide specificities about them. For instance, PT 24's description of student B's solution in the questionnaire illustrated this point:

> Here, the student was able to capture the pattern between the number of steps and the number of rows and reached the correct result, but the student found the solution after many steps.

As the quotation clearly demonstrated, PT 24 provided *a general description* of student B's solution by recognizing the relationship in the pattern.

Lack of Evidence

One prospective teacher's description of student A's solution (3.13%) and three prospective teachers' description of student B's solution (9.37%) were defined as lack of evidence because they either misrecognized or made irrelevant comments on students' thinking. PT 19's explanation related to student B's solution proved this to be true:

> The student's solution is correct. He understood the pattern and expressed it algebraically. He applied the pattern of n. n + 2 to 25^{th} step.

Radford (2008) stated that pattern generalization consisted of three stages: near generalization, far generalization, and writing the rule of pattern. The student B noticed the relationship in the pattern and wrote the 25th step of the pattern based on this relationship. Although student B made a far generalization, he was not able to formulate the rule of the pattern using algebraic expression. Attending is a skill about how the student solves the problem and what he does during the solution phase (Jacobs et al., 2010). Despite the definition of attending, the pre-service teacher described what the student should have done instead of elaborating on the student's current solution. In other words, although student B did not express the pattern algebraically, PT 19 provided the wrong evidence, stating that student B expressed the pattern algebraically. For this reason, PT's explanation was coded as lack of evidence.

Interpreting Students' Algebraic Thinking

This section presents the findings to answer the second research question about prospective middle school mathematics teachers' interpreting of students' algebraic thinking based on students correct and incorrect solutions within the context of pattern generalization. According to Jacobs et al.'s (2010) framework, the second component of professional teacher noticing, which is interpreting students' algebraic thinking, is coded under three categories: robust evidence, limited evidence, and lack of evidence. However, in this study, some participants' responses did not correspond to robust or limited evidence characteristics. Thus, there was a need to add one category, which is named emerging evidence, between robust and limited evidence. Thus, this newly added category describes responses that are not perfect enough to be considered "robust evidence" but perform above "limited evidence." Specifically, *emerging evidence* includes responses in which only **one** of the two expected aspects—either *the correct interpretation of the student's recognition of the relationship* or *the identification of the student's mistake*—is accurately addressed. This category recognizes a partial but meaningful level of interpreting students' algebraic thinking that exceeds l*imited evidence yet* does not meet the full expectations of r*obust evidence*. This revision made it possible to reveal the differences in prospective teachers' interpreting and provide more comprehensive answers to the research questions. The characteristics of the categories related to the dimension of interpreting and the frequency of prospective teachers' responses are displayed in Table 2.

Table 2

Details of the Dimension of Interpreting Students' Algebraic Thinking and the Frequency of Each Category

	Interpreting Students' Algebraic Thinking	Frequency
Student A Solution (incorrect)	*Robust Evidence of Interpreting Students' Algebraic Thinking* Correctly interpreting both the student's exploration of the relationship between the number of squares and the number of rows and columns, and the student's mistake in the generalization of this relationship.	6 (18.75%)
	Emerging Evidence of Interpreting Students' Algebraic Thinking Correctly interpreting either the student's exploration of the relationship between the number of squares and the number of rows and columns or the student's mistake in the generalization of this relationship.	6 (18.75%)
	Limited Evidence of Interpreting Students' Algebraic Thinking Correctly interpreting only the student's incomprehension of the pattern generalization, but the interpretation of the student's algebraic thinking is not in detail.	11 (34.38%)
	Lack of Evidence of Interpreting Students' Algebraic Thinking Making an incorrect or irrelevant interpretation of the student's algebraic thinking.	8 (25%)
	No answers	1 (3.13%)
Student B Solution (correct)	*Robust Evidence of Interpreting Students' Algebraic Thinking* Correctly interpreting the student's exploration of the relationship between the number of squares and the number of rows and columns and the student's generalization of this relationship.	10 (31.25%)
	Emerging Evidence of Interpreting Students' Algebraic Thinking Correctly interpreting either the student's exploration of the relationship between the number of squares and the number of rows and columns or the student's generalization of this relationship.	11 (34.38%)
	Limited Evidence of Interpreting Students' Algebraic Thinking Correctly interpreting only the student's comprehension of the pattern generalization, but the interpretation of the student's algebraic thinking is not in detail.	6 (18.75%)
	Lack of Evidence of Interpreting Students' Algebraic Thinking Making an incorrect or irrelevant interpretation of a student's algebraic thinking.	3 (9.37%)
	No answers	2 (6.26%)

Robust Evidence

Six prospective teachers (18.75%) managed to interpret student A's algebraic thinking with robust evidence, whereas ten of them (31.25%) provided robust evidence to interpret student B's algebraic thinking. PT 7's interpretation of student B's algebraic thinking excerpted from the interview transcript was as follows:

> **Researcher**: What can you say about the student's understanding?
> **PT 7:** Pattern… wait a minute… I looked at the pattern of multiplications. Pattern is actually…

> S/he recognized that the pattern of the number of rows and the number of columns increased one by one in each step and the difference between the number of rows and columns is two in each step.
> **Researcher**: Okay. You said that the student explored the pattern in the questionnaire. How did you make such an inference?
> **PT 7:** The first reason was that the student solved the problem correctly. The second reason was that the student did not write the solution step by step until the 25th step. In other words, after the 11th step, s/he explored the pattern and found it without writing step by step all the steps between. The primary reason for exploring the pattern is to find the result of the far step. Actually, s/he succeeded in here.

PT 7 analyzed that student B properly both *recognized the relationship* between the number of rows and columns, and the number of squares, and then s/he *correctly discovered the pattern*. This response was categorized as *robust evidence* because PT 7 accurately addressed both key aspects: interpreting *the student's recognition of the relationship* and *identifying the student's reasoning in generalizing the pattern*.

Emerging Evidence

Similar to interpreting with robust evidence, six prospective teachers' interpretations of student A's algebraic thinking (18.75%) and eleven prospective teachers' interpretations of student B's algebraic thinking (34.38%) were categorized as emerging evidence. PT 32's interpretation of student B's algebraic thinking in the questionnaire exemplified this claim:

> He knows that he must multiply the [the number of] rows and columns to find the number of squares. Also, he correctly forms a relationship between the number of rows in steps. But he couldn't reach the result. I think there is a lack of attention.

Although PT 32 correctly analyzed that student B could *recognize the relationship* between the number of squares and the number of rows and columns, PT 32 *could not identify the student's mistake* while filling in the table. Therefore, this response was categorized as *emerging evidence* because only one of the two required aspects—*recognizing the relationship*—was accurately interpreted. The failure to *identify the student's mistake* distinguishes it from *robust evidence*, which necessitates the correct interpretation of both elements.

Limited Evidence

Eleven prospective teachers (34.38%) for student A and six of them (18.75%) for student B were able to state whether the student could comprehend the pattern generalization or not, but they failed to refer to the specific points regarding the student's algebraic thinking. For example, PT 3's interpretation of student B's algebraic thinking in the questionnaire portrayed limited evidence:

> This student actually calculated by writing up to figure 11. I think he found the other steps by counting without writing. That's why he set out the figures rather than the concept.

PT 3 only emphasized that the student could solve the problem by focusing on figures, and the prospective teacher could *not provide any details* about the student's algebraic thinking.

Lack of Evidence

Eight prospective teachers (25%) presented misinterpretation and irrelevant comments on student A's algebraic thinking, whereas three of them (9.37%) misinterpreted student B's algebraic thinking and made irrelevant comments on student B's algebraic thinking. PT 19's interpretation of student A's algebraic thinking in the questionnaire indicated a lack of evidence:

> S/he is unable to make sense of the drawing table. S/he made an error while writing the information related to the question on the table. S/he used the table as s/he saw from a friend or from the previous lessons.

PT 19's interpretation *did not include any specific comments* such as the details about the student's recognition of the relationship, their discovery of the pattern, or their generalization. Moreover, this interpretation *was irrelevant to student A's thinking.*

Deciding How to Respond on the Basis of Students' Algebraic Thinking

This section presents the findings to answer the third research question related to the nature of the decisions that prospective middle school mathematics teachers make to respond based on students' correct and incorrect algebraic thinking within the context of pattern generalization. According to Jacobs et al.'s (2010) framework, the third component of professional teacher noticing, deciding how to respond, includes three categories: robust evidence, limited evidence, and lack of evidence. However, in the current study, prospective teachers responded to students by extending/supporting their thinking, reinforcing procedural understanding, or providing a general response. For this reason, it was determined that participants' responses were categorized based on the nature of their responses instead of as robust, limited and lack of evidence. The properties of each category in relation to the dimension of deciding how to respond and the frequency of prospective teachers' responses are presented in Table 3.

Table 3

Details of the Dimension of Deciding How to Respond on the Basis of Students' Algebraic Thinking and the Frequency of Each Category

	Deciding How to Respond to Students	Frequency
Student A Solution (incorrect)	*Extending/Supporting Students' Algebraic Thinking* Supporting student's existing algebraic thinking by asking a question to make the student recognize his/her mistakes/misconceptions.	22 (68.75%)
	Reinforcing Procedural Understanding Asking a similar question with minimal variation (e.g., changing numbers) without supporting the student's algebraic thinking.	1 (3.13%)
	Providing a General Response Asking the question independent from the student's algebraic thinking. Suggesting direct instruction.	8 (25%)
	No answers	1 (3.13%)

	Deciding How to Respond to Students	Frequency
Student B Solution (correct)	*Extending/Supporting Students' Algebraic Thinking* Extending the student's existing algebraic thinking through new questions.	7 (21.88%)
	Reinforcing Procedural Understanding Asking a similar question with minimal variation (e.g., changing numbers) in order to reinforce the student's previously acquired knowledge without extending or deepening their algebraic thinking.	7 (21.88%)
	Providing a General Response Asking the question independent from the student's algebraic thinking. Suggesting direct instruction.	17 (53.13%)
	No answers	1 (3.13%)

Extending/Supporting Students' Algebraic Thinking

A vast majority of the prospective teachers (68.75%) supported the algebraic thinking of student A, who had misconceptions/mistakes, making the student recognize his/her mistakes with follow-up questions. It was surprising that seven of them (21.88%) could extend the algebraic thinking of student B through new questions after s/he solved the problem correctly. For instance, to respond to the student A, PT 7 uttered the following remarks:

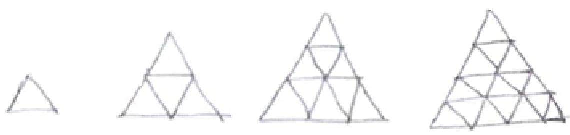
> (1) You said there were 24 squares in the 4th step, and there were 27 squares in the 25th step. How many shapes were there between figure 4 and figure 5? Do you think that the difference between them is three makes sense?
> (2) Can you draw figure 5? Then can you compare the number you found in figure 5 and figure 24?
> (3) (I asked the student to make an estimation.) What has changed in the rows and columns after each step? If the number of rows and columns increases by one, at least how many more squares will there be in figure 5 than figure 4? Can you make an estimation about the number of squares in figure 5? If the number of rows and columns increases by one, what is the difference in number between the number of squares in figure 5 and the number of steps in figure 4? Can you estimate the number of squares in figure 5?

PT 7, in his/her response, *tried to make student A realize his/her mistake via different questions*. In the first question, PT 7 aimed to make student A recognize the fact that there were 27 squares in the 25th step, which was not correct, while the number of squares in the 4th step was 24. PT 7 asked the second question to make student A realize that there were 35 squares in the 5th step, which meant there were more squares than 27. PT 7 supported the student in generalizing the pattern via the third question. Thus, PT 7 supported student A's algebraic thinking.

Reinforcing Procedural Understanding

Only one prospective teacher (3.13%) asked a similar question to student A, who had an incorrect solution without supporting his/her algebraic thinking, whereas seven prospective teachers (21.88%) asked a similar question to student B, who had a correct solution without being able to reinforce his/her algebraic thinking. For instance, to respond to the student B, PT 4 suggested such a question:

"Find the number of triangles in the 25th step of the figure below."



PT 4's question is organized similarly to the question asked on the questionnaire, and it does not force the student to develop a new conceptual understanding and is merely an exercise to develop previously acquired skills and procedures. For this reason, rather than extending student B's thinking, this question encourages student B to consolidate their skills through procedural learning and to apply a particular process more efficiently. For this reason, PT 4's question that was offered to student B is an example of reinforcing procedural understanding.

Providing a General Response

Eight prospective teachers (25%) for student A and more than half of them (53.13%) for student B suggested direct instruction or asked questions that were irrelevant to the student's algebraic thinking. For example, to respond to the student B, PT 18 suggested the following remark:

> Even, student B solved the question correctly. I asked about a similar problem involving different patterns.

PT 18 did *not take student B's algebraic thinking into consideration* and only explained the type of question s/he wanted to direct.

Discussion

Drawing primarily on the "Professional Noticing of Children's Mathematical Thinking" framework suggested by Jacobs et al. (2010), this study aimed to examine prospective middle school mathematics teachers' noticing of students' algebraic thinking on their correct and incorrect responses within the context of pattern generalization. In line with this framework, the findings of the present study were separately discussed under three dimensions: attending to students' solutions, interpreting students' algebraic thinking, and deciding how to respond based on their algebraic thinking.

Attending to Students' Solutions

In this section, the findings related to the first research question about prospective middle school mathematics teachers' attending to students correct and incorrect solutions of pattern generalization were discussed. The findings of the study revealed that more than half of the prospective teachers participating in this study provided robust evidence of attending to students' both correct and incorrect solutions in the context of pattern generalization. The main factor influencing teachers' success in attending might stem from the nature of issues focusing on pattern generalization. As indicated earlier, a pattern generalization process is composed of three stages: (1) near-term generalization, (2) far-term generalization, and (3) writing a rule of pattern (Radford, 2008). To generalize a pattern, students inevitably engage in reasoning in each stage and address the problem following a step-by-step approach (Jurdak & El Mouhayar, 2014; Lannin et al., 2006). Therefore, asking students for a step-by-step solution can aid prospective teachers in identifying how students find a pattern and at which step they make mistakes. Another reason contributing to their success might be related to the properties of the attending skill. Jacobs et al. (2010) defined this skill as the ability to identify how students perform the operations, which tools or figures they use, and how they employ them to represent the key issues presented in the problem. Therefore, the ability to attend to students' responses does not require teachers to identify the conceptual aspects of students' strategies; instead, it requires recognizing the procedural aspects of the strategies implemented. In this respect, this study validated the previous research, which reported that attending, among the three skills, was the one that teachers could apply most easily (LaRochelle, 2018; Sánchez-Matamoros et al., 2019).

Moreover, adding two new categories to the existing categorizations in Jacobs et al.'s (2010) framework provided an opportunity to reveal more clearly the differences and transitions between prospective teachers' attending to students' solutions. The extended categories presented codes and ideas for the researchers to investigate teachers' noticing by using students' incorrect solutions, as well as the correct solutions. Thus, this revision in Jacobs et al.'s (2010) framework contributes to a more meaningful interpretation of the findings related to prospective teachers' attending and more comprehensive answers to the first research question by providing a finer distinction in the analysis. Thus, other researchers can benefit from these categories to investigate teachers' attending to students' solutions within other mathematics contexts with participants from different contexts and backgrounds.

In this section, the findings related to the second research question about prospective middle school mathematics teachers' interpreting of students' algebraic thinking based on students' solutions within the context of pattern generalization were discussed. Similar to attending to students' solutions, it was expected that the step-by-step solution arising from the nature of the pattern generalization process would facilitate teachers' interpretation of students' algebraic thinking. However, it was surprising to find out that the prospective teachers had difficulty in making sense of the students' solutions and interpreting their algebraic thinking. In line with this striking finding, Zapatera and Callejo (2013) found out that some prospective teachers had trouble interpreting students' mathematical thinking in the process of pattern generalization. Another significant finding related to the interpreting skill was that the prospective teachers' success in interpreting students' algebraic thinking largely depended on the correctness of students' strategies. More specifically, it was found that the prospective teachers had more difficulty in interpreting students' algebraic thinking with an incorrect solution than a correct one. Although previous studies pinpointed that the attending skill was the basis of interpretation (Jacobs et al., 2010; LaRochelle, 2018), this result of the study showed that providing robust evidence of attending did not necessarily guarantee robust evidence of interpretation when students' solutions were incorrect.

Attending to students' incorrect solutions requires explaining the details of the correct steps followed, if they exist, and identifying in which step students make the mistakes. However, interpreting the algebraic thinking of students who solve the problem incorrectly requires interpreting the reasoning behind their mistakes. For example, teachers have to elaborate on whether students' misunderstandings are a result of their misrecognition of near generalization or far-term generalization. However, the prospective teachers in this study were not able to explain why the student made the mistakes, although they succeeded in attending to the mistakes. In this sense, it is far from controversial to claim that prospective teachers who could not interpret students' incorrect solutions might not have enough knowledge about students' mistakes/misconceptions. In other words, they might have limited "knowledge of content and students" (KCS), a body of knowledge indicating whether teachers are "be[ing] able to hear and interpret students' emerging and incomplete thinking" (Ball et al., 2008, p. 402).

The extension of Jacobs et al.'s (2010) framework and the addition of a transitional category between "robust evidence" and "limited evidence" in this dimension allows for a more detailed evaluation of the analysis by revealing more clearly the subtle differences in the levels of interpreting to students' algebraic thinking. In particular, the added category of "emerging evidence" allows for a more precise classification of the data and contributes to a more realistic and reliable reflection of the findings. Thus, the responses to the second research question became more comprehensive and nuanced. Finally, the extended categories presented codes and ideas for researchers to investigate teachers' interpreting students' algebraic thinking based on both students' incorrect solutions and the correct solutions. In this regard, these categories to investigate teachers' interpretation of students' thinking within other mathematics contexts with participants from different contexts and backgrounds can be used by other researchers.

Deciding How to Respond on the Basis of Students' Algebraic Thinking

In this section, the findings related to the third research question about the nature of the decisions that prospective middle school mathematics teachers make to respond based on students correct and incorrect algebraic thinking within the context of pattern generalization were discussed. The most striking finding of this study was that the majority of the prospective teachers provided answers that would support the algebraic reasoning of the students, although most of them were unable to interpret the algebraic thinking of the students with incorrect solution. In other words, the teachers aimed to make the students recognize their misconceptions through follow-up questions. On the other hand, collaborating with the results of the previous studies (Crespo, 2002; Milewski & Strickland, 2016), instead of asking divergent questions to the students who solved the problem correctly to expand their algebraic reasoning, most of the prospective teachers gave general responses that were not directly related to the students' thinking. As a consequence, it would be worthwhile to note that the prospective teachers' responses to the students varied according to the accuracy of the students' solutions. To be more specific, the teachers directed more efficient questions to the student with an incorrect solution than the one with the correct solution. These findings allowed us to argue that the prospective teachers having expertise in dealing with students' incorrect thinking might have prior knowledge about students' potential misconceptions, alternative teaching methods for addressing students' misconceptions, and the types of questions that might be asked to make students recognize their mistakes/misconceptions (Milewski & Strickland, 2016). Seen from this perspective, this result suggested that the prospective teachers might have been qualified enough to handle students' misconceptions through their "knowledge of content and teaching" (KCT), indicating that they were familiar with effective teaching strategies and appropriate examples/demonstrations for the teaching of the subject (Ball et al., 2008).

With regard to responding to the student with the correct solution, the prospective teachers may have considered that the task was completed, and thus, they may not have imagined that students' algebraic thinking could be expanded by asking challenging questions. Along with this, they may also have believed that praise was a sufficient response for the students with correct solutions, which stood in parallel with the results of previous research (Crespo, 2002; Milewski & Strickland, 2016). Furthermore, asking problems to extend students' existing knowledge might be challenging for some teachers, as undertaking such a task might require KCT (Jacobs et al., 2010). To explicate further students' correct solutions, teachers should familiarize themselves with effective teaching methods, such as representations and questions that will push the student one step further, which, in a sense, refers to KCT (Ball et al., 2008). Consequently, the differences between the nature of teachers' responses to the students' in/correct solution demonstrated that the prospective teachers had extensive KCT to address the students' incorrect solution, whereas their KCT was relatively limited in terms of explicating on the students' correct understanding.

In addition to KCT, another possible explanation for why prospective teachers did not extend students' correct solutions may relate to their Horizon Content Knowledge (HCK). HCK is defined by Ball et al. (2008) as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). This type of knowledge enables teachers to make informed decisions about how to frame mathematical ideas in ways that anticipate future learning and connect current concepts to more advanced topics (Ball et al., 2008). From this perspective, the inability to expand on students' correct responses may stem from a limited awareness of how the student's current understanding could be deepened or linked to more sophisticated ideas appropriate to their grade level and curriculum. Thus, limitations in HCK may also have contributed to the nature of the prospective teachers' responses, particularly their missed opportunities to extend students' algebraic thinking.

Furthermore, the differences between prospective teachers' responses to the student with correct and incorrect solutions might be related to the content of the Methods of Teaching Mathematics I-II and School Experience courses. In Methods of Teaching Mathematics I-II courses, the prospective teachers may not have focused on how to deepen the understanding of students with correct solutions. Instead, they may mostly have dwelled on how to correct the understanding of students with incorrect solutions. Apart from that, while they were observing the teachers in the classroom as a part of the School Experience course, they may just have noticed students who made mistakes rather than those coming up with correct solutions. Therefore, it can be speculated that prospective teachers were more familiar with offering instructional intervention to students with incorrect solutions as a result of their courses. This could be one of the possible reasons why prospective teachers participating in the current study were far better at offering more effective questions to correct their understanding.

Moreover, the prospective teachers' deficiency in extending the algebraic thinking of the students with a correct solution might be caused by the nature of the pattern generalization process. The problems about pattern generalization are solved by employing a step-by-step method, namely near generalization, far generalization, and writing the rule of the pattern (Jurdak & El Mouhayar, 2014; Lannin et al., 2006). Therefore, it can be a compelling task for teachers to decide on a possible effective intervention with regard to each of these steps for students who offer correct solutions to the problem to extend their algebraic thinking.

Last but not least, the contribution of this study was the different categorization of the third dimension of teacher noticing from Jacobs et al.'s (2010) framework. In this dimension, prospective teachers' deciding how to respond was coded under three categories, which are extending/supporting students' algebraic thinking, reinforcing procedural understanding, and providing a general response instead of as robust, limited, and lack of evidence. This categorization gave an opportunity to examine prospective teachers' responses based on the nature of their responses, which makes this study more sensible. More importantly, it underlined teachers' next steps in terms of supporting and extending students' existing understanding. Furthermore, this modified categorization allowed for the evaluation of teachers' decisions on how to respond to students with both correct and incorrect solutions. To be more specific, prospective teachers' responses to students with correct and incorrect solutions in relation to pattern generalization were granted an opportunity to be discussed together, which in return gave the prospective teachers another perspective related to pattern generalization. Therefore, this categorization might be effective in examining teachers' decisions on responding within other mathematics contexts with participants from different contexts and backgrounds.

In short, although the prospective teachers could attend to students' both correct and incorrect solutions, they had difficulty in interpreting a student's incorrect solution and responding to the students who solved the problem correctly. To put it another way, while attending was an easily practiced skill for dealing with both correct and incorrect solutions, the correctness of students' solutions served as an important indicator of the prospective teachers' interpreting and responding skills. Although the prospective teachers had more difficulty in interpreting the student's incorrect solution than the correct solution, they were more successful in responding to the student who had an incomplete understanding or a misunderstanding in the context of pattern generalization. This finding showed some contradiction with previous research, as it did not completely verify the claim that "deciding how to respond based on children's understandings can occur only if teachers interpret children's understandings, and these interpretations can be made only if teachers attend to the details of children's strategies" (Jacobs et al., 2010, p. 197). More specifically, the findings of the current study showed that although the prospective teachers had the ability to attend to and interpret the students' correct understanding, they found it challenging to respond to the students who had correct reasoning. However, the same relationship between these three skills could not be found for students with incorrect understanding. That is, the prospective teachers had the ability to provide a high level of response to the students with incorrect understanding, though they could not interpret students' incorrect understanding. Based on these findings, this study made significant contributions to the literature on mathematics education by reporting that deciding how to respond to students' incorrect understandings did not require a high level of interpretation of their incorrect understandings.

When considering the limitations of this study, it is noteworthy that the prospective teachers' noticing has been examined solely through two students' solutions. Consequently, there is a need for a more comprehensive investigation in future studies, encompassing a broader spectrum of algebraic topics, and incorporating a diverse set of incorrect and correct student solutions.

Additionally, further studies could be conducted to investigate prospective teachers' noticing of algebraic thinking in other countries by utilizing the categorization presented in the current study. Cross-cultural studies vis-à-vis prospective teachers' noticing of algebraic thinking might be carried out to evaluate whether a cultural dimension of an educational context is an indicative factor for the shape of teachers' noticing. Moreover, investigating the relationship between teacher knowledge and teacher noticing from the point of students' both correct and incorrect understanding within the context of different mathematical domains would be significant as a recommendation for a future study.

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