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# The Genesis of Routines: Mathematical Discourses on the Equal Sign and Variables

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**ABSTRACT**

This study explores high school students' and their mathematics teachers' mathematical discourses on the equal sign and variables in the classroom context. The participants in this case study were ninth graders and their mathematics teachers. The data were analyzed in terms of participants' routines, specifically ritual and explorative routines in the classroom context, through a commognitive perspective. Results indicate that teacher discourse governed process-based routines that have roots in elementary grades. Students focused more on operational approaches instead of algebraic reasoning while solving equations, and this finding conveys challenges in constructing variables.

*Keywords:* equal sign, variables, routines, high school

## Introduction

Equal signs and variables are essential in international and national K-12 curricula and mathematics education research (Ministry of National Education (MoNE), 2018; National Council of Teachers of Mathematics (NCTM), 2005). Students are exposed to the equal sign and variables beginning in the early grades (primary school) and in most mathematical topics (e.g., operations, ratios, first-order equations, and functions) throughout the later grades. The equal sign's pervasiveness at all levels of mathematics emphasizes its significance (Stephens et al., 2013). The equal sign and variables are fundamental components of algebraic thinking and reasoning, which enable students to consider mathematical concepts in an objectified manner instead of as an arithmetic problem that needs to be solved (Kieran, 2004). Kieran (2004) recommends focusing on algebraic thinking and not just correct calculations when working with numbers and variables. To develop students' algebraic thinking, Kieran also suggests tasks that involve comparing algebraic expressions to determine equivalence and investigating the meaning of the equal sign.

Studies have focused on elementary mathematics teachers' knowledge of student thinking on core algebraic concepts, such as the equal sign and variables (Asquit et al., 2007). Among students' struggles when dealing with equations that include unknowns is the concept of variables, which are an integral part of equations, especially starting in the elementary grades. Further, variables play a substantial role in mathematics education research (Küchemann, 1978; MacGregor & Stacey, 1997; Philipp, 1992; Usiskin, 1988). These studies' findings indicate that students encounter challenges using literal symbols in algebra. Küchemann (1978) explored students (13-, 14-, and 15-year-olds) who recognize literal symbols as objects. In contrast, few interpret such symbols as unknown numbers with a fixed value as specific unknowns, a generalized number representing multiple values, or variables representing a range of numbers. Moreover, interpreting letters as variables involves systematic value changes (Küchemann, 1978). In sum, understanding equivalence and variables is essential to algebra, along with the ability to use these concepts with algebraic thinking and reasoning (Knuth et al., 2005). It is significantly more typical for young pupils to perceive the equal sign operationally, as a command to perform computations, rather than relationally, as a signal of equivalence or sameness (e.g., Kieran, 1981; McNeil & Alibali, 2005a; Renwick, 1932). Researchers have shown that using the equal sign operationally can reveal difficulties with algebraic reasoning and errors when solving equations with missing values (McNeil & Alibali, 2005a; Powell, 2012).

Given these challenges for students regarding variables and the equal sign, students may hesitate to join classroom discourse. It is daunting for students facing difficulties to join conversations with their teachers and peers. Without participating in classroom discourse and discussing mathematical concepts, students may struggle to move from ritual routines to explorative routines. Rituals are process-oriented routines, representing the first step in joining a classroom discourse and entail rigidly following a previously performed procedure. On the other hand, explorations are outcome-oriented and self-oriented flexible routines (Lavie et al., 2019). To fully comprehend and help students overcome their challenges, teachers have a vital role. So, relating students' thinking to teachers' teaching from a socio-cultural perspective can provide us with important insights. By extensively considering the literature, researchers have separately explored students’ and teachers’ understanding of equations and variables through cognitive perspectives, mainly in elementary and middle grades. Kieran (2004) has suggested that a more comprehensive analysis, than that reported in the existing literature, is needed to generate rigorous and thorough findings. Little is known about classroom discourse as a means to understand students’ thinking by relating it to teachers’ instruction about equations and variables. Although several studies on equations and variables are based on elementary and middle grades (Asquit et al., 2007; Carpenter & Levi, 2000; Kieran, 2004; Schifter, 1999), few studies have focused on the high school level. This study investigates classroom discourse on the equal sign and variables at the high school level. Thus, our research question is: What are the characteristics of high school students and their teacher’s mathematical discourses on the equal sign and variables in the classroom context?

## Theoretical Framework

### Operational View to Relational View

The operational view has a process-oriented structure such as “doing the operation,” “adding the numbers,” and “calculating the answer.” On the other hand, the relational view expresses equivalence in an objectified manner (Stephens et al., 2013). Skemp (1976) summarized operational conception as a rule without any reasoning.

The equal sign is fundamentally a relational symbol that denotes the similarity or interchangeability of the quantities that either side of an equation represents (Carpenter et al., 2003; Fyfe et al., 2020). Research has indicated that it is significantly more typical for young pupils to perceive the equal sign operationally, as a command to perform operations, rather than relationally, as a signal of equivalence or sameness (Kieran, 1981; McNeil & Alibali, 2005a). Students face challenges transitioning from the operational view to algebraic thinking (Asquit et al., 2007; Bednarz et al., 1996; Kieran, 1992; Wagner & Kieran, 1989). Kilpatrick et al. (2001) observed that students consider the equal sign to be a left-to-right and/or right-to-left directional signal. Similarly, researchers have found that algebra in elementary grades is mainly based on equation manipulation instead of algebraic reasoning (Asquit et al., 2007; Carpenter & Levi, 2000; Kieran, 2004; Schifter, 1999). Steinberg et al. (1990) concluded that most students do not interpret the meaning of two given equations that are equivalent; instead, they focus on solving equations as an operation. Researchers have argued that treating the equal sign operationally can convey challenges in algebraic thinking and mistakes in solving equations with missing numbers (McNeil & Alibali, 2005b; Powell, 2012). Researchers have also mentioned that students should interpret the equal sign from a relational view, that considers the symbol to mean the same, rather than from an operational perspective, that considers the symbol as an arithmetic operation (Falkner et al., 1999; Jacobs et al., 2007; Kieran, 2004; Rittle-Johnson & Alibali, 1999; Knuth et al., 2005; Sherman & Bisanz, 2009).

The overall findings show how deeply ingrained the operational view is and how crucial it is to keep investing effort into supporting students' development of a relational understanding (Stephens et al., 2013). It is important to encourage teachers to see challenges regarding equation tasks as opportunities to use and develop an understanding of the equal sign, rather than just as a chance to practice typical equation-solving methods (Asquit et al., 2007). Studies have suggested that it is vital to discuss both the operational view and the relational view of the equal sign with students while performing equation tasks. Visually supported tasks can encourage students with limited early algebraic knowledge, who often use the equal sign as computing, to interpret the equal sign and equations relationally (Stephens et al., 2013).

To achieve a profound depth of conceptual knowledge, students need a practice that goes beyond memorization of rules and processes. There is a need for professional development initiatives that concentrate on building relationships between middle school algebra instruction and concepts once thought of as belonging to the area of arithmetic (such as understanding the equal sign and developing number sense; Asquit et al., 2007). Understanding how students think about variables and equal signs is crucial for teachers, and expanding their understanding will make it possible for them to focus more closely on students' struggles and communication in relation to the equal sign and variables (Asquit et al., 2007).

**Signs, Symbols, and Objects**

By nature, mathematics comprises mathematical signs, symbols, and objects. Brousseau (1997) argued that the primary pedagogic goal of mathematics instructors' symbolic practices is to communicate mathematics. Mathematical signs and symbols are the main sources for characterizing mathematical knowledge, communicating mathematical arguments, and performing and generalizing it (Steinbring, 2006). While teaching and learning mathematics, we need to use mathematical signs as an instrument for communicating with other people, ourselves, textbooks, and curricula (Sfard, 2008; Vygotsky, 1987). It is therefore fundamental to understand the importance of signs and their relationship with mathematical objects. A sign can be a mathematical symbol, statement, expression, or object in the context of mathematics education (Berger, 2004). According to Tachieli and Tabach (2012), a mathematical object exists between symbols rather than within any of them; hence, no mathematical object can be defined through a concrete object. Students can engage with the mathematical object and communicate with other participants in the discursive community to develop mathematical ideas using mathematical signs (Berger, 2004).

**Commognitive Perspectives**

I utilized the commognitive perspective in this study. “Commognition” is a hybrid word that is a combination of “communication” and “cognition” (Sfard, 2007). The commognitive view formulates thinking as self-communication, and this formulation eliminates the dichotomy between thinking and communication (Sfard, 2008). Thinking is the activity of communicating with oneself (Sfard, 2012) and mathematics can be seen as a discourse, a particular type of communication (Sfard, 2008). Participants do not have to communicate to be part of the same discourse community; however, participation in communication activities enriches participants’ sense of belonging to the broader discourse community (Sfard, 2007). According to commognition, mathematical discourse can be identified through the use of mathematical keywords; visual mediators that refer to graphs, diagrams, algebraic notations, and figures; routines that are repetitive patterns; and endorsed narratives that are substantiated by other elements of discourses (Sfard, 2008). An endorsed narrative is "regarded as reflecting the state of affairs in the world and labeled as true" (Sfard, 2008, p. 298). For instance, endorsed narratives are theorems, definitions, and lemmas in mathematics (Sfard, 2008).

Routines are defined as a set of meta-rules that explain a repetitive action (Sfard, 2008), and a routine is a known pattern of action in a task situation (Lavie et al., 2019). In this study, I focus on examining routines concerning rituals and explorations. Rituals are "sequences of discursive actions whose primary goal (closing conditions) is neither the production of an endorsed narrative nor a change in objects, but creating and sustaining a bond with other people" (Sfard, 2008, p. 241). In a ritual, participants align with other participants in routines in the community, which can be considered social approval (Berger, 2013). Regarding exploration routines the "goal (closing condition) is the production of [an] endorsed narrative" (Sfard, 2008, p. 298). Explorations are routines encompassing solving equations, proving a mathematical result, or generating and investigating a mathematical conjecture (Berger, 2013). For example, during ritual participation, the learner will ask themselves, "How do I proceed?" In explorative participation, the learner will inquire, "What is it I want to get?" (Lavie et al., 2019).

Routines involve not only procedures (the course of action) but also tasks, so analyzing the patterns in a task situation is essential (Lavie et al., 2019). Completion of routines indicates circumstances constituting successful completion of the performance, including how the routines ended in a task situation (Sfard, 2008). Explorations and rituals differ mainly by the types of tasks and their closure (Sfard, 2008). Students' previous experiences establish precedence and heavily influence the routines they use in a new task situation; further, learning occurs through the routinization of students' actions (Lavie et al., 2019). Sfard (2008) asserted that ritual is an inevitable stage in routine development, and new mathematical routines, which are rituals, may evolve into explorations. The participant who does not have a clear idea of when a routine can be implemented may eventually be capable of implementing it independently (Sfard, 2008).

A commognitive perspective provides a lens to understand communicational bindings on mathematical concepts between people and/or materials. Catching communicational failures provides us a perspective on students' understanding and teachers' teaching. If mathematically inconsistent concepts are found in communication, which is labeled as communication failures, the reasons for communicational failures should be explored to comprehend the discourse of the community in general and student discourse in particular. Communicational failures provide a means to catch students' struggles with specific mathematical concepts. Communicational failures are generally observed in rituals because rituals are process-oriented routines to obtain social approval.

Nachlieli and Tabach (2019) presented a methodological lens about ritual-enabling and exploration-requiring learning occurrences. Initiations and closures show when, and the mechanism for how, a routine occurs (Sfard, 2008). Ritual-enabling learning opportunities are described as using a previously recognized procedure. In contrast, the term "exploration-requiring opportunities to learn" is defined as pupils being unable to complete a task just by following a ritual; rather, they need to participate exploratively in developing mathematics narratives concentrating on predicted results (Nachlieli & Tabach, 2019). The teacher will prompt usage of words such as "what," "why," "find," and "explain" during explorative engagement.

In this study, I use Nachlieli and Tabach's (2019) methodological lens to analyze classroom discourse on the equal sign and variables. Analyzing students' routines in classroom discourse enables us to examine the characterization of the classroom discourse. Having investigated students' discourses within their context, I understand its features by relating them to teaching and interaction in the classroom. Due to the nature of this investigation, we can provide suggestions for teachers' discourses as facilitators of students' discourses on the equal sign and variables.

## Methodology

### Context of the Study

Constructivist perspectives have governed the mathematics curriculum in Turkey (Zembat, 2010); however, the primary teaching approach in Turkey is still direct instruction (Emre-Akdoğan et al., 2018). Learning outcomes for equations and variables start in middle school according to Turkey's education curriculum, specifically Grade 6. At this grade level, students are expected to comprehend algebraic expressions. The letters in algebraic expressions represent numbers and are defined as variables (MoNE, 2018). Algebraic expressions, equity, and equations are learning domains in Grade 7. Students should understand the concept of equity, subtract and add algebraic expressions, and solve first-order one-variable equations. In Grade 8, algebra occupies a more significant part of the curriculum, comprising algebraic expressions, linear equations, and inequalities. Students are expected to recognize algebraic expressions and algebraic factor expressions. The foundations of algebra are laid in Grades 6 and 7. Fundamental concepts, such as equations and variables, have a more prominent place in the middle grades. In Grade 9, students are expected to solve first-order one-variable equations. Every ninth grader in Turkey has followed the same curriculum, with specific learning outcomes.

### Participants and Data Analysis

I collected data in an urban school in Turkey's capital from a class of 32 students. In comparison to other ninth graders, the students' success rates were average. The participants in this case study were ninth grade, 15-year-old, high school students and their teacher, Mrs. Seda (a pseudonym). I selected Mrs. Seda's class for our study because she was willing to participate in the research, and she communicated her own and her students' experiences expressively and reflectively to facilitate purposeful sampling and rich, in-depth data collection (Patton, 2002). The data for the study was collected through two classroom observations, each lasting 45 minutes. I focused on the students' and teachers' utterances and actions while transcribing the classroom observations. Analyzing the participants' utterances and actions enabled us to investigate communication failures in the classroom discourse (Emre-Akdoğan et al., 2018). The observed classes were conducted in the participants' native language and then translated from Turkish to English. The transcripts of the classroom observations include participants' utterances and actions. Data regarding participants' routines, specifically ritual and explorative routines in the classroom context, were analyzed (Sfard, 2008).

As indicated in the Theoretical Framework section, we used Nachlieli and Tabach's (2019) methodological lens regarding ritual-enabling and exploration-requiring learning opportunities. We defined exploration-requiring learning opportunities as explorative routines and studied data on how (procedure) and when (initiation and closure) explorative routines occurred. We defined ritual-enabling learning opportunities as ritual routines and studied data on how (procedure)and when (initiation and closure) ritual routines occurred. A routine comprises three parts: initiation, procedure, and closure. The conditions under which a procedure is invoked and by whom, as well as the conditions under which a procedure is regarded as complete, are referred to as initiation and closure, respectively (Nachlieli & Tabach, 2019). Details regarding Nachlieli and Tabach's (2019) methodological lens can be found in Table 1.

According to Miles and Huberman (1994), intercoder reliability enables researchers to give "a clear, unified picture of what the codes mean" (p. 64). First, I independently coded all transcripts, and then a researcher with competence in cognition theory was invited to code the classroom observation transcripts. The number of agreements and discrepancies between the author and the researcher was counted during coding cross-checks. As a measure of intercoder reliability, the ratio of the number of agreements to the number of agreements plus disagreements was utilized (Miles & Huberman, 1994). We achieved 90% intercoder reliability.

**Table 1**

*Ritual and Explorative Routines (Nachlieli & Tabach, 2019)*

| | | Ritual routines | Explorative routines |
|---|---|---|---|
| **Initiation (Task)** | What question does the teacher pose (raise)? | How do I proceed?<br>How can I enact a specific procedure? | What is it I want to get? |
| **Procedure** | How is the routine procedure determined? | Students are expected to follow a certain procedure that others in similar circumstances have previously used. They are not expected to make independent decisions. | Students are expected to choose from a set of procedural options. They are expected to make independent decisions. |
| **Closure** | What type of answer does the teacher expect? | A final solution | An indication of the newly produced narrative |
| | Who determines the end condition? | The teacher | The student (based on mathematical reasoning) |

## Results

In this study, I focused on the teacher's and students' discourses on variables and equations in the classroom context at the high school level. During the two classroom observations, the teacher and students worked on 16 tasks, and the students clarified questions on the tasks and the topic of first-order, one-variable equations. I analyzed the participants' routines according to initiations, procedures, and closures (Sfard, 2008). I observed that the teacher primarily asked the students to perform tasks in the classroom. I investigated the routines based on the tasks the teacher assigned and the students' clarification of the questions regarding their challenges in the classroom. I observed five routines during the classroom observations on equations and variables within the tasks that the teacher assigned in the classroom (Table 2).

**Table 2**

*Types of Routines Identified Through Classroom Observations*

| Routines | Tasks |
|---|---|
| Ritual 1: Left-to-right and/or right-to -left directional signal | TASK1, TASK6, TASK7 (closure: unclear for students), TASK8, TASK9, TASK10, TASK15 |
| Ritual 2: Cross-multiplying | TASK6, TASK7, TASK8, TASK9, TASK10, TASK11, TASK13 |
| Ritual 3: Cancelling factors (e.g., cancel x cubed) | TASK 2 (closure: unclear for students), TASK3 (closure: unclear for students) |
| Ritual 4: Deciding on a solution using an equation (If - 15 = 5, then the solution set is empty; if 0 = 0, then the solution set comprises infinite numbers.) | TASK5 (closure: unclear for students), TASK14 (closure: unclear for students), TASK15, TASK16 |
| Ritual 5: Moving backwards by encircling | TASK10 (closure: unclear for students), TASK11 (closure: unclear for students) |

How can we differentiate between ritual routines and explorative routines? Performing a ritual routine usually entails replicating someone else's previous performance; the procedure is strictly followed, and the performer rarely attempts to make individual decisions. In an explorative routine, on the other hand, students engage in discourse to create new narratives or decide amongst various options (Nachlieli & Tabach, 2019). Depending on the initiation and closure of the practiced routine, the same procedure might be regarded as a ritual routine or an exploration routine (Nachlieli & Tabach, 2019). In this study, the routines had an operational structure that I explored in the classroom discourse, based on the processes labeled as rituals. Additionally, participants rigidly followed the procedure the teacher presented in class and did not engage in individual decision making.

During the classroom observations, the teacher's discourse was governed by two rituals (Rituals 1 and 2), namely left-to-right and/or right-to-left directional signal and cross-multiplying. These routines had an operational structure based on the relevant processes. The teacher initiated the task, and as seen in Table 3, the students rigidly performed the procedure. Closure was attained with the provision of the answer to the task problem the teacher assigned. The following are excerpts from the classroom discourses on Ritual 1, left-to-right directional signal, and Ritual 2, cross-multiplying (Table 3).

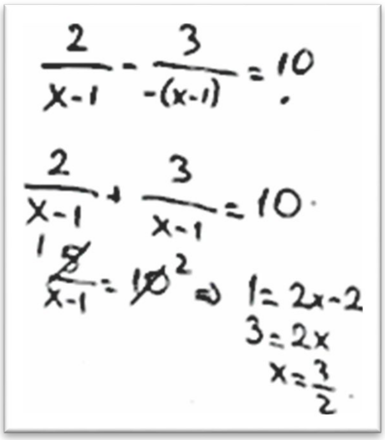
**Table 3**

*Ritual 1 (Left-to-right and/or Right-to-left Directional Signal) and Ritual 2 (Cross-multiplying)*

| | | |
|---|---|---|
| **Initiation** | Teacher: | Solve this equation $\frac{2}{x-1} - \frac{3}{1-x} = 10$. What should we do? |
| **Procedure** | Student 1: | Can we cross-multiply? |
| | Student 2: | By multiplying. |
| | Teacher: | There is no multiplication here. Yes, you can speak [*pointing to another student*]. |
| | Student 2: | We can multiply one of them (*indicating one of the fractions*) with a negative sign, then cross-multiply. |
| | Teacher: | Multiplying one of them with a minus sign—why? |
| | Student 2: | To equalize the denominators. |
| | Teacher: | Can I write it like this to equalize the denominators? [*starts doing calculations*] What did I do with this denominator? I bracket the minus sign [*bracketing the minus sign to the fractional* $\frac{3}{1-x}$], right? $1 - x$ means $x$ - 1 with the minus bracket. Multiply minus with minus; what will it be? |
| | Student 1: | Plus. |
| | Student 4: | Is the answer 4? |
| | Teacher: | Now, I can add with the same denominator. Are the denominators equal? Five over $x$ - 1 is equal to 10. |
| | Student 5: | Cross-multiplying. |
| **Closure** | Teacher: | $2x - 2$. We throw this here [*showing the other side of the equation*]. What happened? 3? 3 is equal to $2x$, and $x$ is equal to $\frac{3}{2}$ . See Figure 1. |

**Figure 1**

*Solving the equation* $\frac{2}{x-1} - \frac{3}{1-x} = 10$



The students became accustomed to working on left-to-right and/or right-to-left directional signal and cross-multiplying rituals during their elementary grades. However, they had no previous experience with the other rituals (Rituals 3–5). Hence, their struggles with Rituals 3–5 were revealed during the classroom observations. The genesis of the routines may have an explorative structure; however, the teacher's discourse included operational explanations of the routines based on the rules. By observing the routines' closure, I found that the students' discourse was unclear for these routines.

The canceling ritual (Ritual 3) includes a process-based interpretation of variables' canceling factors, that is, a particular way of defining the parameters of first-order differential equations to simplify terms. As seen in Table 4, closure was reached with the provision of the answer to the task problem the teacher assigned. I will provide a classroom discourse initiated by a teacher-assigned task (Table 4).

**Table 4**

*Ritual 3. Canceling Factors*

| | | |
|---|---|---|
| **Initiation** | Teacher | If the given equation is a first-order equation dependent on $x$, find the solution set of the equation [ $(a-2)x^3 + (b-3)x^2 + 4x + 2a + 4b = 0$]. |
| **Procedure** | Student 2: | Do we try to cancel $x^3$ and $x^2$? |
| | Teacher: | Yes, cancel $x^3$ and $x^2$. This is the third degree; this is the second degree; then, what will happen to them? [*showing* $x^3$ *and* $x^2$] What should it be? $a-2$ should be equal to $0$; then, $a$ is equal to $2$. $b-3$ should be equal to $0$; then, $b$ is equal to 3. Then, let's write them in the equation: $(2-2)x^3 + (2-2)x^2 + 4x + 2.2 + 4.3 = 0$ [*writing the a and b values in the equation*]. Here [*showing* $2-2$], what happened to 0? Here [*showing 3*$-3$], what happened to 0 multiplied by 0? $4x + 4 + 12 = 0$, and $4x + 16 = 0$ [the *students repeat the same equation*]. |
| | Student 1: | Put $16$ on the other side of the equation. |
| | Student 1: | 4. |
| | Teacher: | $x = 4$, right? |
| | Student 2: | $x = -4$ |
| **Closure** | Teacher | So, the solution set is $-4$. Yes, where do we use the curly brackets? For the solution sets because they are the sets of points. |

Ritual 4 includes deciding the solution set of the given first-order equation by interpreting a final equation—for instance, if -15 = 5, the solution set is empty, or if 0 = 0, the solution set is infinite—without discussing the mathematical thinking behind these equations. In Task 14, the teacher asked the students the following question: "If the solution set of this equation $m\ (2\text{-}x) = nx + 4$ is infinite, then what is $n$?" This task initiated the classroom discourse, and the student and teacher discourses are given in Table 5. I labeled this routine as a ritual because the procedure was rigidly performed without independent decisions.

**Table 5**

*Ritual 4. Finding the Solution Using an Equation*

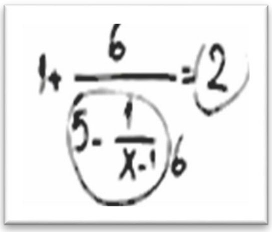
| | | |
|---|---|---|
| **Initiation** | Teacher: | If the solution set of this equation $m(2 - x) = nx + 4$ is infinite, then what is $n$ ? Now, what does it mean to have infinite elements? |
| **Procedure** | Teacher: | So, we write $a$ is equal to 0 and $b$ is equal to 0 for the equation ax + b. If 0 is equal to 0, then what can we say? <br> $a = 0\ b = 0$ <br> $ax + b = 0$ <br> $0 = 0$ <br> Infinite elements, so the solution set for this equation is real numbers [*writing this Ç.K. = R*], right? Okay, let's organize this: 2*m* minus *mx* minus *nx* minus 4 equals 0. Let's have the bracket of *x*, -*m* minus *n* plus 2*m* minus 4 equals 0 [ *x (- m - n)* + 2*m* – 4 = 0]. Here is the expression with *x* [*showing the coefficient of expression with x*]. What will be the coefficient of the term with $x$? [*There is no answer from the students*]. It needs to be 0 : 0 multiplied by *x* plus 0 equals 0. Okay. What will this be [*showing* 2*m* – 4]? Here, it will be 0. So, - *m* - *n* is equal to 0; from here, if I take $n$ to the other side of the equation, *m* is equal to - n [ - *m* – *n* = 0 => *m* = - *n*]. 2*m* minus 4 is equal to 0, and then *m* is equal to 2, right? [2*m* – 4 = 0 => *m* = 2]. If *m* is equal to 2, then what is *n*? |
| **Closure** | Teacher: | $-2.$ [$\left.\begin{matrix}-m-n=0 \Rightarrow m=-n \\ 2m-4=0 \Rightarrow m=2\end{matrix}\right\} n=-2$] <br> [$m(2-x) = nx + 4$ <br> $a = 0\ \ b = 0$ <br> $ax + b = 0$ <br> $0 = 0$ <br> $Ç.K. = R$ <br> $2m - mx - nx - 4 = 0$ <br> $x(-m-n) + 2m - 4 = 0$ <br> $\left.\begin{matrix}-m-n=0 \Rightarrow m=-n \\ 2m-4=0 \Rightarrow m=2\end{matrix}\right\} n=-2$] |

In the given classroom discourse, the teacher's discourse led the students' discourse on Ritual 4, which had a procedural structure comprising the following: i) If you find one type of equation (-3 = 5), then the solution is empty; ii) if you find another type of equation (2 = 2), then the solution is infinite. However, the relationship between an equation and a solution set was not so thoroughly discussed in the classroom discourse as to make the discourse transparent for students. Questions such as "Why is there a relationship between an equation and a solution set? "What is a solution set?" and "What is an empty or infinite solution set?" need to be clarified in the classroom discourse.

In Ritual 5, moving backwards by encircling entails solving an equation starting from the back and moving toward the front. While performing the procedure, participants encircled some portion of the whole equation that the teacher presented. For instance, when the teacher presented Figure 2 as the given equation, the participants encircled the denominator of the equation.

**Figure 2**

*Encircling the Denominator of the Equation*



In Ritual 5, the teacher initiated the classroom discourse by assigning Task 10: Find the value that satisfies $x$ in the equation:

$$1+\frac{6}{5-\frac{1}{x-1}}=2$$

See Figure 3 for the procedure and solution and Table 6 for the discourse.

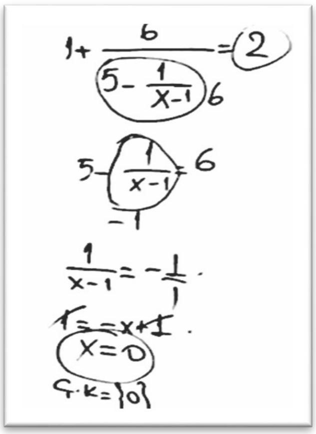
**Table 6**

*Ritual 5. Moving Backwards by Encircling*

| | | |
|---|---|---|
| **Initiation** | Teacher: | Find the value that satisfies *x* in the equation. $1+\frac{6}{5-\frac{1}{x-1}}=2$ (Task 10) |
| **Procedure** | Student 1: | Balloon. |
| | Teacher: | Balloon. Okay, you named this a balloon. |
| | Student 2: | Balloon. We call this an equation. |
| | Class: | This is a balloon. |
| | Student 3: | We call this an unsolvable equation. |
| | Teacher: | In this type of task, we start solving at the end [*indicating the denominator* $5-\frac{1}{x-1}$]. However, I did not solve this task like this [*indicating the denominator* $5-\frac{1}{x-1}$]. So, what is the product? [*indicating* 2]. |
| | Class: | 2. |
| | Teacher: | 2. Okay. What should I add to 1 to get 2? [*indicating* 1] |
| | Student 1: | 1. |
| | Student 2: | 1. |
| | Teacher: | Okay, [*closing 1*], cancel this; what will be here? [*indicating the fraction* $\frac{6}{5-\frac{1}{x-1}}$ ] |
| | Student 1: | 1. |
| | Teacher: | To have 1 here, what will be the value of here [*indicating* $5-\frac{1}{x-1}$ ]? |
| | Student 1: | 6. |
| | Teacher: | 6 [*writing* $5-\frac{1}{x-1}=6$ ]. 5 minus 1 over *x* - 1 equals? |
| | Student 1: | 6. |
| | Teacher: | 6. So, what is the value here? - 1? [*encircling the minus* 1 *over x minus* 1] |
| | Student 1: | Yes. |
| | Student 2: | Yes. |
| | Teacher: | - 1 and 5, what is at the front? $a$ minus sign [*encircling* $\frac{1}{x-1}$]. 5 minus 1 is 6 [*indicating the equation*]. So, what is the value of $\frac{1}{x-1}$? [*writing* $\frac{1}{x-1}=-1$]. |
| | Student: | - 1. |
| | Teacher: | - 1. Now, by cross-multiplying, 1 is equal to negative *x* plus 1. We put 1 on the other side of the $\frac{1}{x-1}=-\frac{1}{1}$ $1=x+1$ $x=0$ equation, so *x* equals 0. If we write *x* equals 0, what will be the answer? |
| **Closure** | Teacher: | The answer is 2 (Figure 3). We write 0, then what happens? Minus 1 minus 1 minus becomes plus 5, plus 1 6, 6 over 6 1, 1 plus 1, 2, so it is true [*by substituting* 0 *for x into the equation and controlling whether the equation is satisfying*], right? By substituting, I am controlling for if I found the correct answer. Thus, the solution set is [*writing on the board Ç.K.* = 0 *(ÇK is the abbreviation of the solution set in Turkish)*]. Is this challenging? |
| | Student 1: | Yes. |
| | Student 2: | Yes. |
| | Class: | Yes, challenging. |
| | Student 3: | It is not challenging; it is annoying. |

**Figure 3**

*Solution of TASK10*

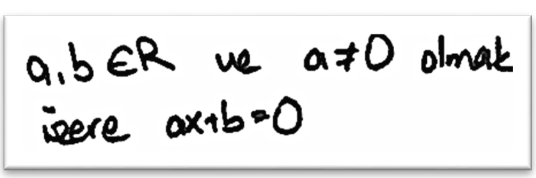


In the given classroom discourse, the students encountered challenges completing the task, the teacher's discourse was dominant, as shown in the excerpt, and the students mostly tried to imitate the teacher’s discourse. One of the significant points of this classroom discourse is the closure of students’ discourses, which included “This is challenging” and “annoying and long.” Such statements indicate the struggles that the students encountered. Moreover, the students’ discourse on variables revealed struggles. Below is a classroom discourse on variables.

| | |
|---|---|
| Teacher: | When I say first-order equation, there needs to be 1 [*indicating the exponent of x*]. Figure 4 is a first-order equation. How many variables are there? |
| Student 1: | 3. |
| Student 2: | 2. |
| Student 3: | 1. |
| Student 4: | 3: *a, x*, and *b*. |
| Student 5: | No, *a* and *b* are natural numbers, so there is just one variable. The others [*a, b*] represent numbers. |
| Teacher: | She said just one. So, here is just one variable; what should I call this? A first-order one-variable equation. Okay? So, what are *a* and *b*? They are real numbers. |

**Figure 4**

*Example of First-order Equation*




In the classroom discourse, the students considered *a, x*, and *b* as variables because they are unknown in the given equation. The students also considered all unknowns as variables. The teacher stressed that *a* and *b* are real numbers but did not mention the definition of a variable. A variable can also be a real number that can be changed at once in a range of numbers. Thus, the teacher's discourse on variables was not apparent to the students. After implicit classroom discussions about variables, I found a communication failure regarding the difference between letters as variables and unknowns (*x*, *a*).

Teacher: … If the value that satisfies $x$ for the equation $\frac{1}{x+1} - \frac{2}{x+a} = 1$ is -2, what is *a*? How can I ask this question another way?
[*There is no answer from the students.*]
Teacher: If the root of the given equation is -2, what is *a*?
Student 1: But it needs to be by the *x* variable, right?
Teacher: It is by the *x* variable, but there are no other variables in the equation.
Student 1: There is *a*.
Student 2: *a* is a number.
Student 1: When the root of an equation is mentioned, it needs to say that *x* is a variable.
Teacher: There is no variable except *x*.
Student 3: *a* is a number; *a* is a number!
Student 1: Ha, okay!

In the given classroom discourse, the teacher considered *x* to be a variable and *a* to be an unknown. However, differentiating between variables and unknowns was still challenging for the students, so they considered both *x* and *a* to be variables. The questions "What is an unknown?" and "What is a variable?" had to be clarified for the students. At that point, realization of a mathematical definition and statement comes to the forefront.

Another challenge in the classroom discourse was the "root of an equation." A student asked about understanding the root of an equation. The classroom discourse on this topic is given below.

Student 1: What is the meaning of the root of an equation?
Teacher: What is the root of an equation? What is the root?
Student 2: *x*.
Student 3: *x*.
Student 4: Variable.
Teacher: So, is it *x*? If the root of an equation is - 1?
Student 5: Result.
Teacher: So, what is the value of *x*? If it is equal to *x*?
Student 6: Solution set.
Teacher: What is the solution set? - 1. Let's look at the solution sets in the other tasks.
Student 1: *a* is 5, right?
Student 3: *a* is 5.
Teacher: Okay, listen to me. We found that *x* is equal to 3, okay? [*indicating the solution of the empty set*] All of these [*indicating the solution* $x = \frac{3}{2}$]. The root of these equations [*indicating the answers to the three tasks*].
Student 1: Root.
Teacher: So, satisfying this value [*indicating the equation*]. Please take note if you do not know this. Satisfying the value means the root of an equation. What do we call the value that satisfies *x*? The root of the equation.

The students interpreted the root of the equation as a variable and an unknown; thus, it represents *x,* which, for them, can be a variable and an unknown. One of the students explicated *x* as a solution set. Using this approach, the teacher interpreted the root of an equation as a value that satisfies the equation. Another challenge for the students was finding the solution set in different number systems. This result may be attributed to Ritual 4, a conditional process-oriented routine that includes deciding the solution set if 5 = 3 or 2 = 2. Below is a classroom discourse based on the teacher-assigned task: Find the solution of $2x - 11 = 0$ in *N, Q, Z,* and *R*.

Teacher: …What does this mean?
Student 1: Natural number, whole number.
Student 2: Natural number.
Teacher: Find the natural number, whole number, rational number, and real number. Okay, find the solution [*The students are working on the task*].
Teacher: Yes, what is the answer in *N, Z*, and *Q*?
Student 1: 5 in *N*, 5 in *Z*, $\frac{11}{2}$ in *Q*, $\frac{11}{2}$ in R.
Teacher: So, you found 5 in *N*, 5 in *Z*, $\frac{11}{2}$ in *Q* and *R*; any other answers?
Student 2: 6 in *N* and *Z* because it becomes 5, 5, when we round up.
Student 3: I found 6.
Teacher: Congratulations! Any other ideas?
Student 4: $\frac{11}{2}$ in *Q* and *R*. I could not find any solutions in *N* and *Z*.
Student 5: Can we say 5, 5 in real numbers?
Teacher: Yes, you can say it in real numbers. Is 5, 5 not an element of the real number? Or $\frac{11}{2}$. So, what was our question? $2x - 11 = 0$, $x = \frac{11}{2}$. Whose element is $\frac{11}{2}$?
Student 6: Rational and real.
Student 7: Rational and real numbers.
Teacher: So, real numbers and rational numbers. As you know, real numbers subsume rational numbers.
Student 1: Yes.
Teacher: The answer in real and rational numbers is $\frac{11}{2}$. However, if I ask you for the solution set in *N*, the solution set in *N* and *Z* is an empty set. If you say, "It is $\frac{11}{2}$, so I round up and take 5 or 6," this is impossible. You cannot make up; you cannot round up! Can the answer be rounded up? Substitute 6 in the equation; does it satisfy? Substitute 6, 2 multiplied by six minus 11 equals 0. Is this right?
Student 1: Nope, it is equal to 1.
Teacher: So, what happened? I round up and take 5; rounding up and taking 6 is impossible. What do you round up? This is impossible. So, what do we say for the solution set in *N* and *Z*?
Student 1: Empty set
Teacher: Empty set. What is the solution set in rational numbers in *Q*?
Student 1: $\frac{11}{2}$
Teacher: $\frac{11}{2}$. The solution set in *R* is $\frac{11}{2}$. Okay?

According to the classroom discourse, the students' recognition of the solution set depended on the operational view of rituals. When the students encountered different tasks requiring an explorative realization, they struggled to interpret the solution set and navigated based on their discourse. In conclusion, the root of an equation is unknown and variable.

## Discussion and Conclusions

This study explores the mathematical discourses of high school students and their mathematics teachers on the equal sign and variables in the classroom context. The results indicate that the teacher's teaching was governed by ritual teaching, in which she had the students strictly follow the procedures she performed. Additionally, the teacher enabled students' provision of short, closed questions as the first step to join a new discourse. Ritual teaching may provide a foundation for explorative teaching and is the first step in supporting students' expansion of their mathematical discourse based on previous learning experiences (Nachlieli & Tabach, 2019). Aligning with earlier research (Nachlieli & Tabach, 2019), our findings suggest that ritual instruction is essential for both object- and meta-level learning because it serves as a foundation for explorations and assists students with their first steps into a new discourse. If the teacher assigns tasks from the operational perspective, student discourse can be consistent and dominant in the operational view. For instance, the students imitated the teacher's discourse. In this study, the students used rituals with imitating teacher's discourse to gain their teacher's social approval. Practicing rituals is a first step to get into a discourse of the mathematics classroom, however, students can have challenges interpreting the equal signs and variables independently, on their own, in a flexible way. Besides, when students face tasks that include explorative characteristics, they might have difficulty interpreting the teacher's discourse. Students may struggle when the teacher's discourse is not transparent to them (Emre-Akdoğan et al., 2018). The characteristics of initiations, tasks and closures are among the essential components of routines' genesis. Aligned with Nachieli and Tabach's (2012) findings, I explored the strong relationship between tasks and ritual routines. Additionally, rituals are strongly associated with very restrictive tasks (Sfard, 2008). In this study, the students were accustomed to working on left-to-right and/or right-to-left directional signal and cross-multiplying rituals with which they were familiarized during their elementary grades. However, they had no experience with the other rituals. Thus, I observed their struggles with Rituals 3–5 in the classroom discourse. This may be because of the explorative structure of the routines' genesis that emerged from the tasks. However, the teacher's discourse included operational explanations of the routines based on the rules. This led to challenges for the students and unclear student closure of the routines. Presently, different rituals cannot be seen as interchangeable since they produce the same closures (Sfard, 2008).

In line with the literature, I explored students' perception of the equal sign as a left-to-right and/or right-to-left directional signal (Kilpatrick et al., 2001). Students in this study focused more on operational approaches than algebraic reasoning and mathematical objects while solving equations (Asquit et al., 2007; Berger, 2004; Carpenter & Levi, 2000; Kieran, 2004). According to the Turkish curriculum, the foundations for algebra are laid in Grades 6 and 7. Fundamental concepts, such as equations and variables, are more prominent at the middle-grade level. Subsequently, in Grade 9, students are expected to solve first-order, one-variable equations and consider equations from an operational view. However, the literature stresses that if students do not have a relational understanding of the fundamental concepts, they will struggle to solve equations (Kieran, 2004). Further, suppose students do not thoroughly understand the sign by relating it with mathematical objects. In that case, their use of mathematical signs will be restricted, and they may face challenges while implementing them (Berger, 2004).

Student discourses do not have a clear approach to unknowns, and this may be due to the fact that students tend to use ritual routines as imitating the teacher's operations in the class. This approach to unknowns conveys struggles with basic mathematical constructs, such as variables and the root of an equation. One of the reasons for students' struggles with unknowns may be due to teacher discourse on unknowns and variables, where the teacher implements process-based routines that do not have an explorative structure. As Berger (2004) mentioned, students' use of signs within a social community allows them to develop the sign's meaning, which is compatible with the community. The other reason may be students' preexisting routines on the concept of unknowns. Also, just using ritual routines with unclear closures convey struggles on the concept of unknowns. If, during the elementary grades, students consider unknowns to be a tool to perform operations as ritual routines, and do not have a relational understanding in an explorative way and relate unknowns with mathematical objects, this may lead to challenges. In line with the literature, this study found that the teacher's discourse on using letters as specific unknowns, generalized numbers, or variables was not transparent and explicit for students (Küchemann, 1978). This led to students' difficulty differentiating between variables and unknowns in this study. Küchemann (1978) stressed that mathematics teachers use the blanket term "variable" to refer to any letters in generalized arithmetic. However, to improve algebraic thinking, it is important to be aware of using letters as objects, specific unknowns, generalized numbers, or variables in algebra (Küchemann, 1978).

In conclusion, both ritual and explorative routines should be used in the classroom to create more explicit and transparent discourses for students. To diversify the types of routines implemented in the classroom, task characteristics play a substantial role. Teachers should assign operational and explorative tasks in the classroom. These types of tasks should convey operational and explorative routines. Moreover, it is necessary to conduct longitudinal research that extensively explores at which level, when, and how to implement operational and explorative routines. Moreover, mathematics educators should focus on different performances in a broader discursive context to analyze routines in detail since the difference between ritual and exploration lies in when they are conducted (Sfard, 2008).

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