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General Proof Tasks Provide Opportunities for Varied Forms of Reasoning about the Domain of Mathematical Claims

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ABSTRACT

Proofs are attempts to conclusively demonstrate the validity of the claim for all cases indicated within its domain, which implies that proving should involve thoughtful consideration of the domain. This study analyzed the enactment of three general claim tasks, or tasks where the domain of the claim referred to an infinite number of cases, that were used during an introduction-to-proof teaching experiment with 10 ninth grade students. We analyzed the tasks in terms of the opportunities students experienced to engage in reasoning-and-proving and attend to the domain of the claims. The use of general claim tasks provided students with opportunities to engage in varied reasoning-and-proving activities, including forms not typically found in textbooks. Students' attention to the domain of the claims increased over the course of study as a result of the teacher-researcher's continued focus on this aspect of the tasks, although their attention did not always encompass all cases within the domain. By making the domain of mathematical claims a central focus, we emphasize its important role in the reasoning-and-proving opportunities afforded to students and contribute to an understanding of students' early interpretations of this aspect of proof tasks.

Keywords: reasoning and proving, instructional intervention, secondary mathematics, geometry

Introduction

The types of reasoning-and-proving tasks given to students impact their learning opportunities and shape the way in which they are able to reason about the mathematical content. With respect to reasoning-and-proving tasks used in high school geometry, students' opportunities are influenced not only by the validity of the claim, but also by the number of cases indicated within the domain of the claim (a single case, multiple but finitely many cases, or infinitely many cases) (Stylianides & Ball, 2008). For instance, claims involving a single case (e.g., prove a given triangle ABC is congruent to triangle DEF) can be essentially proven or certainly disproven by measuring the given sides and angles, whereas claims involving infinitely many cases (general claims) provide an intellectual necessity for a deductive approach[1]. In other words, since general claims cannot be proven using examples (Buchbinder & Zaslavsky, 2019), they are particularly well-suited to motivate the need for deductive reasoning (Otten, Gilbertson et al., 2014).

In addition to motivating deductive reasoning, general claim proof tasks in high school geometry courses can allow students to consider fundamental mathematical ideas. Because a proof of a general claim eliminates the possibility of counterexamples, it can result in the prover's ability to say with absolute certainty that a mathematical statement is true for all cases within the domain of the claim (e.g., Ellis et al., 2012; Fischbein, 1982). This is not to say that the production of a proof necessarily *convinces* a student the claim is always true (Rodd, 2000), just that it offers a level of certainty not afforded by examples. The ability to know with absolute certainty that a claim is always true is one way that mathematics (and physics) distinguishes itself from the biological sciences (Schoenfeld, 2000). Additionally, the use of both general and particular claims allows students to reflect on the number of cases encompassed within its domain, a worthwhile endeavor in and of itself that does not receive sufficient attention (Mason, 2019). Finally, the use of general claims reflects the broader practice of mathematicians who seek to pose and prove conjectures that encompass as many cases as possible.

Given the benefits of general claims, it is unsurprising that they are commonly used in studies focused on secondary students' understanding of proof and their ability to construct proofs (e.g., Buchbinder & Zaslavsky, 2019; Chazan, 1993; Healy & Hoyles, 2000; Knuth et al., 2009). This focus on general claims is not reflected within the reasoning-and-proving opportunities provided in textbooks. Otten, Gilbertson et al. (2014) analyzed U.S. high school geometry textbooks and found that student exercises involve particular claims much more than general claims. The discrepancy between the domain of claims used in proof tasks by researchers and those found in geometry textbooks highlights a need to better understand how the use of general claim tasks potentially impacts students' opportunities to (1) engage in the reasoning-and-proving process and (2) consider the domain of the claims being proven. Thus, this study examined a series of general claim tasks used during an introduction-to-proof teaching experiment with ten ninth grade students in the Midwest United States. Specifically, we examined the learning opportunities of proof-related tasks, as set up by the teacher-researcher, implemented with students and how students attended to the domain of the claims, evidenced in their written and verbal work. By making general claim proof tasks an explicit item of focus, we seek to promote greater understanding of the relationship between the domain of the mathematical claim and students' learning opportunities.

Defining Key Terminology

Reasoning-and-proving is used to refer broadly to all of the activity that goes into establishing the truth-value of a claim, from proposing a conjecture and investigating the validity of the claim, to constructing a proof or providing a justification that does not reach the level of a proof (non-proof rationale) (G. J. Stylianides, 2008). The term *proof* refers to "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" that uses acceptable justifications, valid modes of argumentation, and representations that are appropriate and understood by the classroom community (A. J. Stylianides, 2007, p. 291). Although Stylianides (2007) focused on the classroom community in an elementary setting, in the present study we interpreted the terms "valid", "acceptable", and "appropriate" according to both the classroom community and the broader

[1] There are specific mathematical claims that require deductive reasoning (e.g., prove that $2^{191} - 1$ is prime); however, these claims do not tend to be located in high school geometry textbooks. Within secondary education, claims occasionally fall into a separate category when they ask students to prove a claim for a relatively small number of cases (e.g., for numbers 1-20; see Knuth et al., 2009). For these tasks, students can reasonably check every single example (proof by exhaustion). That said, these claims tend to be numerical and are not typically used in high school geometry courses.

mathematics community because the context of secondary mathematics marked a shift toward formal proving. Additionally, we use the term *proof* as an adjective describing the tasks where students were expected to construct a proof and the term *argument* to refer to students' verbal or written work made in response to a proof task. Note that the term *argument* does not carry judgment about the quality of students' response or the extent it is aligned with Stylianides' (2007) and our definition of proof.

Recall that the *domain of mathematical claims* refers to the number of cases implicitly or explicitly referred to in the mathematical statement or theorem. Fischbein (1982) articulated the important role the domain of claims has in the proving process saying, "The level of generality of the theorem is then explicitly defined by the theorem itself and the proof refers exactly and clearly to that level of generality" (p. 15). In other words, theorems and statements to be proven indicate the domain of the claim (level of generality) and proofs demonstrating the validity of a given claim must clearly demonstrate it for all cases included within the domain of the claim. We continue with Otten, Gilbertson, and colleague's (2014) use of the terms *particular* and *general statements* to distinguish between proof tasks involving claims that reference a single case (particular statements) and those that encompass an entire, often infinite, set of cases (general statements). Geometric proof tasks that fall under the latter category typically use the quantifiers "all", "every", or "a" (i.e., "an arbitrary case") to indicate the domain of the claim.

Theoretical Perspective and Literature Review

Opportunities to Learn

Although there is a large body of literature focused on the teaching and learning of proof, few studies have specifically focused on students' opportunities to learn reasoning-and-proving with respect to the domain of the mathematical claims. Opportunity to learn originally referred to whether students, prior to being assessed, solved mathematical problems similar to those contained in the assessment (Husen, 1967, as cited in Floden, 2002). It is important, however, when determining students' opportunities to learn how to solve a certain type of problem, to disentangle the content topic of the problem and the specific formulation of the problem (Floden, 2002). In other words, it is one thing for students to be exposed to the mathematical ideas necessary to solve a problem; it is another thing for them to have practiced the exact type of problem presented in an assessment. As discussed above, proof tasks involving general claims may deal with mathematical content that is familiar to students, but their prior experiences may have been formulated with particular claims or situated within learning opportunities that did not draw attention to the domain of the claim.

The opportunities to learn framework can be particularly powerful when analyzing mathematics classrooms and instructional interactions given its demonstrated ability to connect teaching and learning (Hiebert & Grouws, 2007). The value of this perspective has emerged since the 1960s as opportunity to learn has come to be framed as more than topic coverage or problem-type familiarity; one can consider the topics together with the level of cognitive demand students' experience (Gamoran et al., 1997), the topics combined with the classroom learning environment (Tarr et al., 2013), or the interactions that occur in the classroom while topics are being taught (Jackson et al., 2013). In this study, we take the latter approach as we move beyond studying opportunities in textbooks (as summarized in the following sections) to a consideration of the opportunities that students have to think and discuss the reasoning-and-proving process *as* they work on tasks. In particular, we analyze the opportunities to engage in the reasoning-and-proving process within the tasks as launched by the teacher-researcher and in students' engagement in the tasks in order to allow for possible differences between the tasks' potential and realized opportunities (Stein et al., 1996).

Students' Attention during Mathematical Tasks

While it is certainly important for students to engage in the desired forms of reasoning-and-proving in order to increase their opportunity to learn this mathematical practice, doing so does not necessarily ensure that the intended learning will occur. Mason (2008) contended that "what teachers *can* do for learners, indeed perhaps the only thing they can actually do *for* learners, is to direct learners' attention [italics in original]" (p. 31). Ingram (2014) agreed, noting that students' attention can be influenced by features of the task and the interactions they have with teachers. Yet, even in instances where students and the teacher are collectively working on a single task, there is a potential for miscommunication to occur due to differences in where their attention is focused (Mason, 2008). When considering students' attention, one can focus on where that attention is directed but also the structure of attention. Structures of attention, according to Mason, include "holding wholes, discerning details, recognizing relationships, perceiving properties and reasoning on the basis of agreed properties" (2008, p. 35).

In order to understand the differences in the way that novices and experts attend to mathematical ideas, Mason and Davis (1988) coined the term "shifts in attention", which they defined as a moment, either sudden or gradual, "in which one becomes aware of what used to be attended to was only part of a larger whole, which is at once, more complex and more simple" (p. 488). During the proving process, students should begin shifting their attention away from specific details in examples or diagrams, towards a focus on generalizing through attention to mathematical relationships (Ellis, 2011). When interpreting mathematical statements being proven, students should recognize that the words *all*, *every*, and *any* indicate the impossibility of an exception that satisfies the criteria in the hypothesis but contradicts the conclusion (Harel & Sowder, 2007). Diagrams play an important role in many geometry proof tasks. Using diagrams when proving a general claim requires the ability to view the diagram as both an expression of generality (that is, a representation of all diagrams indicated within the domain of the claim) *and* as an object that can be manipulated (through rotations, adding notation, axillary lines, etc.) (Mason, 1989). Teachers can interpret diagrams as *figural concepts*, possessing both spatial properties and conceptual qualities (Fischbein, 1993) in part because they have been enculturated into the community of mathematics, wherein attending to the generality of mathematical claims is a central idea. In contrast, students who have not yet undergone this shift in attention may interpret the diagram only through the lens of its spatial properties or other features that are specific to the diagram drawn. Mason (1989) conjectured that "this is precisely where sophisticated mathematician-teachers, unaware of the momentary abstraction in themselves, miss the need to attend to the abstracting movement in their students" (p. 6). While it is important for teachers to attend to the ways that students are interpreting mathematical diagrams, shifting their attention towards the generality of claims is not something that a teacher can do or force onto students (Mason & Davis, 1988), nor is it something that can be achieved solely through calling attention to this aspect of mathematical claims (Mason, 2004). But students' opportunities to engage with general proof tasks may provide the context in which shifts in attention can occur.

Reasoning-and-Proving Opportunities in Textbooks

From the opportunities to learn perspective, students' thinking about reasoning-and-proving and the domain of mathematical claims is influenced by the opportunities embedded in curriculum materials. With respect to the introduction to proof chapters in U.S. Geometry textbooks, Otten, Males, and Gilbertson (2014) found that the student exercises primarily provided opportunities for them to investigate or pose conjectures and develop non-proof rationales, but few opportunities to construct a proof. Given that the introduction to proof chapter occurs early in Geometry textbooks, it makes sense that students' content knowledge might limit the number of proof tasks that are appropriate for the beginning of the school year. However, the limited opportunities to construct proofs suggests that students are developing their initial understanding of proof *without* actually engaging in the proving process. Looking at a random selection of the remainder of the Geometry textbooks, beyond the introductory chapters, Otten, Gilbertson et al. (2014) found that proof opportunities were prevalent, but they predominantly involved particular claims. The general claims that applied to infinite sets of geometric objects were typically presented in the textbook narrative, not the student exercises.

This focus on the domain of the claims is important because, although most reasoning-and-proving textbook studies (e.g., Fujita & Jones, 2014; Hanna, 1999; Miyakawa, 2012; Stylianides, G., 2009) have consistently examined the type of argument elicited (e.g., empirical, generic example, direct proof), it is by identifying the domain of the mathematical claims (i.e., general, particular, or general with particular instantiation) that we can consider whether those opportunities involved claims that *necessitate* a deductive proof. As Otten, Gilbertson et al. (2014) pointed out, deductive reasoning is powerful enough to establish the truth of both particular and general claims, but only deductive reasoning is able to establish the truth for general claims. Thus general claims necessitate deduction to a greater degree than do particular claims. Based on the findings that general claims were relatively rare in student exercises in U.S. Geometry textbooks, Otten, Gilbertson et al. (2014) called for future research analyzing the enactment of these opportunities in order to understand the role that the domain of mathematical claims might play with regard to students' experiences with proof.

Reasoning-and-Proving Opportunities in Classroom Settings

Research on proof instruction in secondary classrooms have primarily described whole-class conversations (e.g., Otten et al., 2017), providing only a snapshot into the reasoning-and-proving opportunities afforded to students in the classroom. Within this setting, teachers tend to spend a significant amount of time in their Geometry classes focusing on the details of proofs, such as whether each "step" in the proof contained a mathematically-correct justification and logically flowed from the previous statements (Martin & McCrone, 2003; Otten et al., 2017; Schoenfeld, 1988). Furthermore, traditional classrooms tend to operate based on specific norms around who is responsible for different aspects of the proving process (Herbst & Brach, 2006). For instance, students are rarely asked to prove their own conjectures (Boero et al., 2007); in instances where students are asked to conjecture, the teacher tends to confirm whether it is correct before students prove the claim (Herbst & Brach, 2006). In contrast to the teacher-driven reasoning-and-proving that occurred in the prior studies, Martin and colleagues (2005) described four classroom episodes where the teacher and students shared ownership in the reasoning-and-proving process. In these episodes, the teacher used revoicing and coaching in order to hold students accountable for contributing to the construction of the proofs. All of the whole-class conversation captured in the aforementioned studies focused on the task at hand (e.g., completing the proof) with little if any conversation that afforded students the opportunity to think broadly about reasoning-and-proving as a mathematical practice (Otten, Gilbertson et al., 2014).

Proof studies in secondary classroom settings have primarily occurred toward the middle or end of the school year; as a result, little is known about how students are first introduced to proof in traditional classrooms. One exception are the studies conducted by Cirillo (2011; 2014), who reported that six teachers introduced proof in their Geometry classrooms through a show-and-tell approach. During the teachers' proof demonstrations, Cirillo noted that the teachers did not explicitly unpack the many different components of proof, such as how they were using definitions to draw conclusions or what can and cannot be assumed from a diagram. In sum, there is still a need to better understand ways to introduce students to proof that utilizes a student-centered approach and develops students' understanding of proof through engaging in the reasoning-and-proving process, especially as those early opportunities relate to the domain of the claims being proved.

Students' Proving in Relation to the Domain of Mathematical Claims

One consistent pattern throughout proof research is the finding that a non-trivial percent of students construct empirical arguments for general proof tasks (Reid & Knipping, 2010; G. J. Stylianides et al., 2017). The construction of empirical arguments for general claims has been documented in studies of middle school students (Knuth et al., 2009), high schoolers (Healy & Hoyles, 2000; Lee, 2016; Senk, 1985), and undergraduate students (Harel & Sowder, 1998). Example use during the proving process is not inherently bad because students can productively use examples to gain insights into why a conjecture is true or uncover structural relationships (Aricha-Metzer & Zaslavsky, 2019). Nonetheless, students' use of examples as justification can reveal challenges in understanding that the goal is to construct an argument that applies for all cases, where no exceptions are possible (Harel & Sowder, 2007).

There are multiple possible explanations for why secondary students tend to produce empirical arguments when proving general claims. First, it is possible students recognize that empirical arguments do not prove general claims, but still write them because they lack the mathematical skills to be able to construct a more general argument (e.g., Bieda & Lepak, 2009; Healy & Hoyles, 2000; Reiss et al., 2001). However, this explanation does not account for instances when students use a few examples as justification in instances when proof by exhaustion would be a valid approach (Knuth et al., 2009). A second possibility is that some students misinterpret or do not recognize the domain of a mathematical claim due to a lack of explicit language indicating that the statement applied to an infinite number of cases (Mason, 2019). Or, it is possible that students are attending to the quantifiers indicating the domain of the claim but interpret them using a "real world" rather than mathematical definition (Pimm, 1987). These potential explanations speak to the importance of scholars not only attending to the empirical or deductive arguments that students produce, but also to their interpretation of the claim's domain.

While students can productively use diagrams as a planning tool or to capture their progress in a deductive argument (Cirillo & Hummer, 2021), others interact with diagrams in ways that suggest an interpretation of the diagram as a specific example (Herbst, 2004). Like Chazan (1993), Martin and colleagues (2005) found that students requested a proof for a second type of triangle even though the claim had been proven for a generic triangle, a request that suggests a lack of realization that the original proof demonstrated the claim was true for all triangles. In both instances, it is possible that the students were attending to generic and specific features of the diagram rather than interpreting it as a generic example. Infrequent opportunities to produce their own diagrams may also contribute to students' limited understanding of how to appropriately interpret a diagram. Although the norm of teachers or textbooks providing diagrams (Cirillo, 2018; Herbst & Brach, 2006) increases the consistency and accuracy of the diagrams students use, it limits their opportunities to reason about what the diagram represents or about the generality indicated within the proof claim (Komatsu et al., 2017).

Research Questions

Collectively, prior research on students' understanding of proof and the ability to construct proofs highlights a need for changes to the ways that proofs are taught in the classroom, particularly in order to fulfill the recommendations that reasoning-and-proving should be a central part of K–12 instruction (Ministry of Education, Science and Technology, 2011; National Council of Teachers of Mathematics, 2009; National Governors Association & Council of Chief State School Officers, 2010). This study extends the textbook analysis of Otten, Gilbertson et al. (2014) by analyzing the enactment of three general claim tasks in terms of the reasoning-and-proving opportunities they afforded, and students' attention to the domain of mathematical claims. The research questions that guided the analyses were as follows:

> *RQ1*. What opportunities for reasoning-and-proving were present in general claim tasks set up by the teacher-researcher and implemented with students during an introduction to proof unit?
>
> *RQ2*. How, if at all, did students attend to the domain of the claims and what, if any, shifts in attention occurred over the course of a task enactment?

Both questions were addressed through an analysis of students' oral and written work on the tasks, with the data and analytic processes described in detail in the next section.

Method

Participants and Data Collection

Ten students participated in this study—seven females (Amanda, Arin, Heather, Lauren, Lexi, Megan, and Sadie) and three males (Brian, Clay, and Wilson; all pseudonyms). They were the only students enrolled in an accelerated ninth grade mathematics course at a rural, public school in the Midwest United States. The accelerated program covered Algebra 1 and 2 content in ninth grade; subsequently, the study provided the students first formal high school Geometry instruction. All sessions were held during the school day but outside of their regular mathematics class. Students received a graphing calculator for their participation.

The exploratory teaching experiment (Steffe & Thompson, 2000) consisted of 14 sessions, held twice a week with each session lasting between 28 and 38 minutes. All sessions were taught by the first author, who is identified as the teacher-researcher (TR) in this article. The use of a researcher as the teacher is consistent with teaching experiment methodology (e.g., Cobb & Steffe, 1983; Steffe & Thompson, 2000) and should not be confused with self-study methodology, wherein the researcher studies their own teaching in classrooms where they are the main instructor. Every session was video and audio recorded to ensure that students' gestures, manipulation of physical objects, and voices during small-group discussions could be reviewed, with one audio and video recorder placed near each group. Additionally, all written work and students' responses to journal prompts were collected during the sessions. For this study, audio/video recordings served as the primary data source; students' written work and journal reflections were referenced as needed in order to provide a more complete picture of what occurred during the sessions.

Teaching Experiment Design and Rationale

Exploratory teaching experiment methodology is used to study students' ways of understanding and operating with particular content in instances when testing the researchers' hypotheses for learning may not be appropriate (Steffe & Thompson, 2000). In particular, this methodology was selected in order to better understand students' ways of understanding proof while engaging in tasks that are not commonly found in traditional classrooms. The primary goal of the present study was to develop students' understanding of the purpose of proof through their engagement in tasks that emphasized the proving process as a means of a) developing certainty that the given statement is *always* true and b) understanding *why* it is always true (de Villiers, 1990; Hanna & Jahnke, 1996). Specifically, we hypothesized that the explanatory feature of proofs could help students transition away from empirical arguments since examples on their own do not tend to explain why a statement is true. We chose to only use tasks involving general claims based on the hypothesis that they could facilitate student understanding that a proof must contain justifications that encompassed all objects within the claim's domain, particularly when accompanied by conversations where the domain was an explicit object of focus.

The TR structured the instruction so that students developed their understanding of proof as they engaged in various reasoning-and-proving activities. The goal to engage students in authentic reasoning-and-proving has been used in a variety of intervention-based studies (e.g., G. J. Stylianides et al., 2017). For example, the present study's use of general claims and having students prove their own conjectures was successfully used in a study with eighth graders (Boero et al., 1996). Finally, we incorporated statements *about* reasoning-and-proving (Otten, Gilbertson et al., 2014) into whole-class discussions and through the use of reflection prompts in order to focus students' attention on specific aspects of proofs. Although hypotheses for learning were developed to guide the instruction, additional iterations of the teaching experiment would be needed in order to test and revise the instruction so that students' ways of understanding aligned with the researcher's hypotheses (or the hypotheses could be revised upon further iterations). During the sessions, the TR did not focus on the form of proofs, but instead allowed students to write arguments in a way that made sense to them.

The primary tasks used in this study were developed before the start of the experiment based on the hypotheses described above. On the other hand, the time spent on each task and select sub-tasks were devised during the study in response to where the TR interpreted students to be in their current understanding. For example, the original tessellation task, "do all quadrilaterals tessellate?" was pre-planned, but the follow-up task, "do all regular polygons tessellate?" was added mid-experiment in an attempt to focus students' attention on both the sides and angles of polygons. We describe the three focal tasks in the following sections; see Appendix A for a description and rationale for an overview of the entire instructional sequence.

Overview of the Tessellation Tasks

After reviewing the definition of quadrilaterals and introducing tessellations, the TR launched the tessellation task by posing the question, "Do all quadrilaterals tessellate?" Students were given six sets of different convex quadrilaterals to aid in their investigation. At the end of the session, the TR asked students to journal how confident they were that quadrilaterals always tessellate and to describe how they would explain their answer to a friend. In Session 2, students were asked to write a set of "step-by-step" directions for how to tessellate *any* quadrilateral. Each group was given some of the quadrilaterals from Session 1 as well as two concave and one convex quadrilaterals to use during the task. In Session 3, the TR introduced the next subtask by asking students if they knew of other polygons that they thought would always tessellate. After eliciting their ideas, the TR introduced regular polygons and referenced familiar examples. Next, the TR posed the question, "Do all regular polygons tessellate?" Regular polygons were selected because they fit within students' directions for tessellating quadrilaterals, despite only some tessellating. In order to investigate this question, the TR first provided all groups with a set of regular hexagons and then passed out regular pentagons, septagons, and octagons (one per group) to "speed up" the process. The reflection prompts provided in Session 3 (see Appendix A) were used to encourage connections across the tasks and motivate a need to understand *why* quadrilaterals always tessellate. The term "counterexample" and the idea that only one counterexample was needed to disprove a general claim was introduced to students towards the end of Session 3. The TR concluded the tessellation task in Session 4 by summarizing the key ideas from the first three sessions and then explaining why quadrilaterals and regular hexagons, but not regular septagons or octagons, tessellate.

Overview of the Constructing Quadrilateral Diagrams Task

Prior to launching the diagrams task, the TR briefly introduced students to conditional statements and demonstrated how they are used in the proving process by talking through an informal proof of the conditional statement, "If a quadrilateral has 360°, then it will tessellate."[2] Afterwards, students worked in three small groups to draw diagrams for statements 1–3 during Session 7 and statements 4–6 during Session 8 (Table 1). Since they had not yet taken high school Geometry, the theorems were rephrased to exclude potentially unfamiliar terminology such as "congruent," "consecutive angles," and "supplementary." At the end of each session, the whole class discussed specific features of the diagrams, including the different notation methods they had used.

Table 1

Statements Used in the Constructing Diagrams for Quadrilateral Theorems Task

1.	If the polygon is a rectangle, then the diagonals have the same length.
2.	If a quadrilateral is a parallelogram, then the measures of the angles on the same side of the shape add to 180 degrees.
3.	If a quadrilateral is an isosceles trapezoid, then the diagonals have the same length.
4.	If two sides of a parallelogram that intersect have the same length, then the parallelogram is a rhombus.
5.	If the diagonals of a parallelogram form a 90-degree angle, then the parallelogram is a rhombus.
6.	If one angle of a parallelogram is a right angle, then the parallelogram is a rectangle.

Overview of the Proving Similar Polygon Conjectures Task

The TR introduced the similar polygons task by asking students to pose conjectures of specific polygons that they thought might be similar (i.e., "all ____ are similar"). During the launch, they discussed the teacher-posed conjecture "all polygons are similar" to make sure that students understood the conjecture and remembered how to use a counterexample to disprove a conjecture. Students posed four conjectures that involved classes of polygons that were always similar to one another: squares, equilateral triangles, right triangles, and rhombuses. After constructing their argument for the conjecture "all squares are similar", small groups exchanged papers and provided feedback to their peers. The TR posed the following questions to focus students' feedback: "Is it convincing? Does it convince you that no matter what two squares I draw, they're going to be similar? And is there anything someone could say to poke a hole in the argument?" Next, each group revised their own argument in response to the two sets of peer feedback they received. In Session 12, the TR led the entire class through the proof of the conjecture, "all squares are similar", which built on elements that were in students' arguments from the previous session. Afterwards, students investigated the classes' conjectures for right triangles and equilateral triangles and then constructed an argument demonstrating that the conjecture was either true or false. Students investigated the final conjecture, "all rhombuses are similar", during the final interview and then constructed an argument either proving or disproving it, depending on their belief in the conjecture's validity.

Analytic Process

In addition to rooting our study in the literature previously described and explaining the relationship between the researchers and study participants, we now articulate our process in the data

[2] This claim was stated by one of the students in an earlier session. We chose to use their phrasing in order to connect to the student's earlier words instead of starting with a claim that was more mathematically precise. During the class discussion, the TR clarified that the hypothesis referred to the angles of a quadrilateral.

reduction and analysis process in order to make claims using qualitative research methodology (Noral & Talbert, 2011). We restricted our analysis of the data to sessions involving geometric tasks. There were five broad geometric tasks: the tessellation tasks (Sessions 1-4; 103 minutes), constructing diagrams task (Sessions 7-8; 68 minutes), constructing a definition for similar polygon conjectures task (Sessions 9–10; 62 minutes), proving similar polygon conjectures task (Sessions 11–12; 157 minutes[3]), and proving the exterior angle theorem task (Session 13; 34 minutes). The unit of analysis was a response (written and/or verbal) to one question or prompt (subtask) within the identified tasks. Specifically, units of analysis spanned the time between when student(s) started and completed each subtask in whatever grouping configuration they were placed. Most subtasks were completed in three small groups; however, six subtasks (four reflection and two math prompts) were completed individually. We excluded whole-class discussions in instances when they only reiterated students' small group work so as not to double analyze reasoning-and-proving activity. Collectively, there were 119 units to be analyzed. In this article, we present findings for the tessellation, constructing diagrams, and proving similar polygon conjectures tasks since they best illustrate the range of reasoning-and-proving that occurred.

To answer RQ1, we analyzed the session data using the qualitative research software MAXQDA to determine the opportunities students had to engage in reasoning-and-proving based on the launch and implementation of each task. In order to make comparisons between the types of reasoning-and-proving opportunities found in regular Geometry textbooks and the instruction used in this study, we adapted the expected student activity portion of Otten, Gilbertson et al.'s (2014) analytic framework, which was a modified version of Thompson and colleagues (2012)'s framework (see Table 2).

Table 2

Reasoning-and-Proving Student Activity Codes

Related to Mathematical Claims	Related to Mathematical Arguments	Emergent Codes
• Make a conjecture, refine a statement or conjecture, or draw a conclusion • Fill in the blanks of a conjecture • Investigate a conjecture or statement	• Construct a proof • Develop a rationale or non-proof argument • Evaluate an argument/proof • Find a counterexample	• Make sense of a mathematical claim • Construct a diagram • Revise an argument/proof

Note. The codes in the first two columns are from the framework described by Otten, Gilbertson et al. (2014).

We adapted Otten, Gilberston et al.'s (2014) framework such that it applied to both the anticipated reasoning-and-proving activity and the reasoning-and-proving students actively engaged in during the sessions. For example, in the exchange below during the proving similar polygons task, both students' comments were coded as *develop a rationale or non-proof argument*. Additionally, the entire exchange was included within a broader *evaluate an argument/proof* code to capture the broader reasoning-and-proving activity being completed.

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[3] Students individually completed the final prompt of this task during the final interview; this accounted for 93 of the 157 minutes.

Wilson: I don't think [the angles] should be [labeled] A, B, C, D, I think it should be A, A, A, A cause they're all the same angle

Megan: and then they need those [notation] on the edges [sides] of the square to show that it's the same length cause that's what makes it a square.

Develop a rationale or non-proof argument was used in instances when students provided a justification for a single statement ("because…") and in instances when students were asked to "explain" or provide a justification for "why" a claim is true. In contrast, *construct a proof* was used in instances when the task directly asked students to prove a mathematical claim. Note that the presence of this code did not guarantee that the resulting product contained all of the required elements to be considered a proof. During the coding process, we identified additional instances of reasoning-and-proving that occurred in the sessions but were not captured by Otten, Gilbertson et al.'s (2014) codes. This resulted in three additional codes: *make sense of a mathematical claim*, *construct a diagram for a mathematical statement*[4], and *revise an argument/proof*. *Make sense of a mathematical claim* was used in instances when students talked about a claim without trying to determine whether it was valid (the latter would be coded *investigate a conjecture or statement*). For example, Arin's second statement below, which occurred during the constructing diagrams for quadrilateral theorems task, was coded *make sense of a mathematical claim* because she was not trying to actively determine if the claim was true.

Arin: *(reading)* "If a quadrilateral is a parallelogram" is that the like one that's straight lines and then it like *(makes a slanted line gesture with her hands)*

Sadie: Yeah

Arin: Okay. *(reading)* "then the measures of the angles on the same side of the shape add to 180." Yeah, because one angle is going to be bigger than the other.

Instances when students discussed mathematical vocabulary, such as Arin's first sentence, were not coded as *make sense of a mathematical claim* because it was activity outside of the reasoning-and-proving process. Finally, the code *revise an argument* referred to instances when students revised their draft argument in response to peer feedback. To increase trustworthiness (Lincoln & Guba, 1985), the authors had continual calibration conversations with one another and also produced preliminary analytic memos that were vetted by an outside observer.

To answer RQ2, we first analyzed students' discussions during each task in order to assess how, if at all, students were attending to the domain of the claims. Although it is not possible to ascertain what students were internally attending to at a particular moment in time, we could look for evidence of shifts in attention through what they said or did in their conversations with peers (Barwell, 2002, as cited in Ingram, 2014). Examples of students demonstrating attention to the domain of the claim include a student responding to a peer's assertion by saying, "no that's not true for all of them". Next, we analyzed students' written work for evidence of attention to the domain of the claims. Specifically, we determined whether students' justifications, constructed diagrams, and notation methods encompassed all cases within the claim's domain. Although we coded students' written work as indicating attention to the domain of the claim, or a lack thereof, we recognize the possibility that a student could construct a general argument or notate their diagram with variables for reasons other than their understanding of the claim's domain. Additionally, it is possible that a student could

[4] Constructing a diagram could have been coded using the "modify or revise a mathematical statement" code since students were adding a diagram to accompany the provided statements. However, a new code was added to emphasize the fact that textbooks and teachers rarely, if ever, hold students responsible for producing a diagram for a proof task (Cirillo, 2018; Herbst & Brach, 2006).

recognize the domain of the claim but produce an argument that only refers to a finite number of cases.

After coding for individual instances of attention to the domain of claims, we then looked across the coded data for evidence of shifts in attention (Mason & Davis, 1988). For example, if a group of students initially labeled their diagram with specific angle or side measurements in one instance, but later labeled them with variables, this would indicate a shift in attention to the fact that the mathematical claim being represented by the diagram refers to an infinite class of quadrilaterals. All instances where the students' attention to the domain of the claim was unclear were discussed with an outside observer; in these instances, we include possible alternate interpretations in the results.

Findings

We describe students' engagement in three general claim tasks in terms of the reasoning-and-proving opportunities that surfaced during the tasks. We then share students' interpretation of and attention to the domain of each mathematical claim. The findings are structured by task in order to (1) highlight the range of reasoning-and-proving afforded within a single task, and (2) acknowledge that the mathematical content of the task (Dawkins & Karunakaran, 2016), other features of the task, and its location in the instructional sequence may have influenced students' attention to the domain of the claims. Across the three tasks, students engaged in all of the reasoning-and-proving opportunities set forth in the launch of the tasks as well as additional, unplanned reasoning-and-proving that arose during small-group and whole-class conversations. Although students demonstrated limited attention to the domain of the claims at the beginning of the tessellation task and constructing diagrams task, there were small shifts in attention by the end of both tasks. In contrast, during the proving similar polygon conjectures task, students attended to the domain of the claims throughout their conversations and through their written justifications. However, some students attended to the domain of the claims in a way that did not encompass all possible cases during the rhombus portion of the similar polygons task.

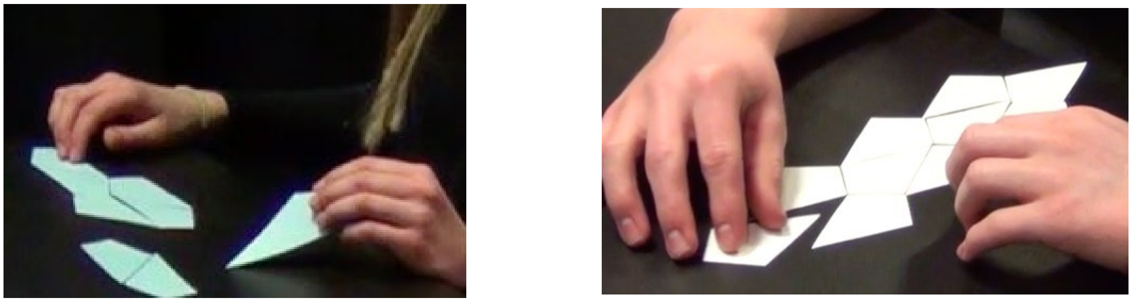
Tessellation Tasks

Varied Reasoning-and-Proving Opportunities

The tessellation tasks, as launched by the teacher-researcher (TR), provided students with opportunities to investigate the validity of mathematical statements, construct a counterexample, and develop non-proof rationales for their assertions. Specifically, Session 1's subtask ("do all quadrilaterals tessellate?") resulted in non-proof rationales as students investigated how to tessellate different quadrilaterals in their small groups (Figure 1) and again as they individually summarized their responses in their notebooks. In Session 2, the subtask to create "how to" directions for tessellating *any* quadrilateral did not explicitly provide opportunities for students to engage in reasoning-and-proving, but did encourage greater attention to the domain of the mathematical claims.

Figure 1

Two Students Tessellating Different Irregular Convex Quadrilaterals in Session 1.



Session 3 involved two subtasks; the first, "do all regular polygons tessellate?", allowed students to investigate a conjecture and find a counterexample. The second subtask ("Do you still think that all quadrilaterals tessellate? If no, explain why. If yes, is there something special about quadrilaterals that make it so that they will always tessellate?") provided opportunities for non-proof rationales. In addition to engaging in all of the intended reasoning-and-proving activity, students also made conjectures, posed counterexamples in response to a peer's conjecture, and refined a peer's conjecture. The additional reasoning-and-proving activities occurred in Session 3 while students discussed as a whole class the possible characteristics of polygons that tessellate.

While investigating the validity of the claim ("do all quadrilaterals tessellate?"), students' non-proof rationales in their written reflections at the end of Session 1 referenced the different cases that had successfully tessellated and an assumption that the pattern would continue to hold true for other cases. For example, Sadie wrote, "I'm very confident that all quadrilaterals tessellate. Since we tested out many different shapes and they all worked, it helps prove my point. I would convince [a friend] by showing them how I found out that they all fit." Similarly, Amanda wrote, "all quadrilaterals tessellate because if you match up one side of the quadrilateral, the other sides will have to match up too." Across students' written work at the end of Session 1, nine students included non-proof rationales to justify why they thought all quadrilaterals tessellate.

The final subtask around why all quadrilaterals, but not all regular polygons, tessellate produced the most varied opportunities for reasoning-and-proving, including activities that were not requested in the original prompt. After writing down their justifications, students discussed in small groups and then as a whole class possible reasons why only some polygons tessellate. The dialogue below occurred during the whole-class conversation.

Amanda: I said that maybe after a shape gets like, like after they have four sides, like five and on, then maybe the angles become too wide, because they have too many sides

TR: Okay. What do y'all think about that?

Wilson: Well, hexagons work, but…

TR: So hexagons work… Lexi, can you speak up a little bit?

Lexi: Okay, well I said maybe. *(Arin quietly interrupts her)*

TR: Go ahead [Lexi] and say what you were thinking.

Lexi: Okay, well I said maybe like after four sides the sides have to be even with the amount, cause five didn't work.

TR: Five didn't work, yeah; so that would be a reason for that…but what, hold on, will you talk a little bit louder?

Arin: The octagon didn't work.

In this exchange, Amanda and Lexi posed conjectures that described features of polygons they thought would tessellate (or not) and Wilson and Arin responded to each claim with a counterexample the class had previously investigated. The fact that hexagons tessellate was a counterexample to Amanda's idea that polygons with more than four sides could not tessellate, and octagons failing to tessellate was a counterexample to Lexi's idea that even-sided polygons might tessellate. The discussion is notable given that students had not yet been formally introduced to the use of counterexamples in the proving process or the idea of revising a claim. Instead, the additional unplanned reasoning-and-proving activity surfaced as students discussed two general claims (do all quadrilaterals tessellate? And do all regular polygons tessellate?) that were similar in structure but differed in validity. Through the use of two such general claim tasks, students were not only able to investigate the validity of the claims and provide non-proof rationales, but were also able to pose conjectures and counterexamples by looking across the two tasks.

Increased Attention to the Domain of the Claim

Throughout Session 1, students' justifications relied on a lack of counterexamples rather than the identification of specific features of all quadrilaterals that result in them tessellating. Thus, there was little explicit attention on the domain of the claim. Students' attention to the domain of the claim increased during Session 2 as they developed a series of "how to" directions for tessellating any quadrilateral. For example, Megan and Arin's written directions stated: "1st we put opposite angles together. 2nd we repeated the first step as well as flipped and mirroring the shapes from the original two shapes. Same side length, different angles". These directions represent a shift in attention from haphazardly moving copies of a quadrilateral around until they "fit" to purposefully placing the quadrilaterals together by focusing on the sides and angles. When describing how to tessellate any quadrilateral, Megan and Arin referred to generic sides and angles and did not mention more specific features that some, but not all, quadrilaterals contain (such as 90 angles or congruent sides). The reference to generic features of quadrilaterals may have been a result of the task prompt to create step-by-step directions for how to tessellate *any* quadrilateral rather than a change in how students were interpreting the different provided examples, but regardless, the shift in attention was noteworthy.

Small-group conversations in Session 3 revealed students' varying attention to the domain of the claim. For example, Clay suggested they put two quadrilaterals together so that they make a nicer shape such as a rectangle or square, and Heather responded by saying, "that's just for this shape, it's not for all of them. Like different quadrilaterals make different shapes, not just a square." Although Clay appeared to be focusing on features of certain quadrilaterals, Heather's response suggests that she was considering multiple quadrilaterals when thinking about how to place the two copies together to form a tessellation. In the whole-class discussion around features of polygons that determine whether they will tessellate in Session 3, both Amanda and Lexi's justifications referenced general features of polygons (the number of sides and angles) rather than specific characteristics. Students' increased use of statements that applied to multiple if not all quadrilaterals in Sessions 2 and 3 suggest at least some attention to the domain of the claims. Given the explicit emphasis on *all* or *any* when launching the subtasks, it is possible that students' use of these words in their conversations reflected the instructional focus rather than how they were mentally thinking about the claims (Mason, 2004). Nonetheless, students' written work and conversations revealed moments where at least some students appeared to be considering multiple, if not all possible cases.

Constructing Diagrams for Quadrilateral Theorems Task

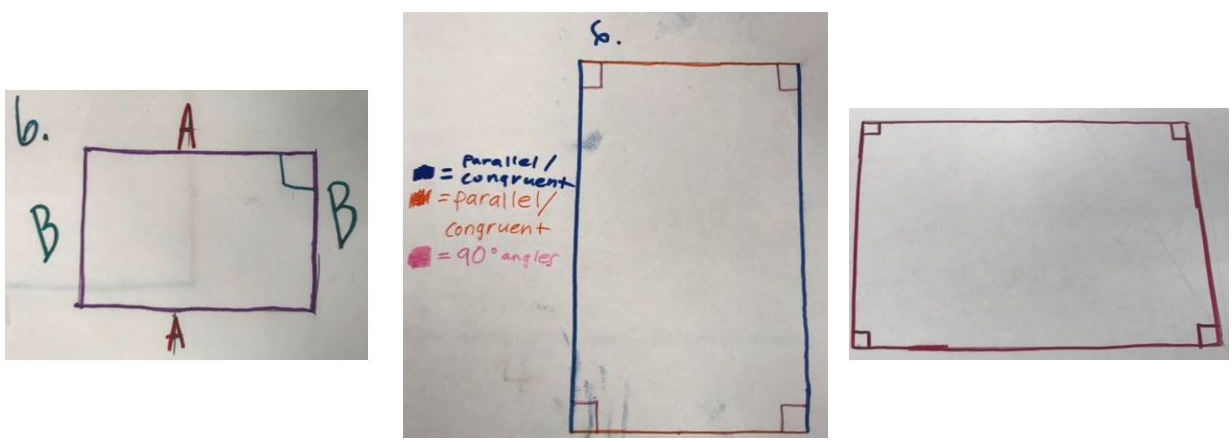
Limited Reasoning-and-Proving Opportunities

The constructing diagrams task, as launched by the TR, provided students with the opportunity to construct a diagram for six quadrilateral theorems (see Table 1 for task directions and Figures 2–4 for examples of student-constructed diagrams). In addition to engaging in the intended reasoning-and-proving activities, students also informally drew conclusions and made sense of the mathematical claims. For example, the three small groups' diagrams for the theorem, "if one angle of a parallelogram is a right angle, then the parallelogram is a rectangle" are shown in Figure 2.

As students constructed the diagrams, some began informally drawing conclusions from the hypotheses by applying their prior knowledge of quadrilaterals. While constructing the left diagram in Figure 2, Wilson argued that all of the angles had to be 90 degrees based on the given information. "This one has to be 90 degrees since...they're all 90 degrees, yeah. Because these (adjacent angles) have to add to 180 and if one of them is 90 degrees, the other has to be 90 degrees." Even though the task did not ask students to construct proofs for the given theorems, it allowed the opportunity for students to begin informally reasoning about the theorem and verbally begin to draft a rough outline for a mathematical argument.

Figure 2

The Three Small Groups' Diagrams for the Theorem, "If One Angle of a Parallelogram is a Right Angle, then the Parallelogram is a Rectangle."



Note. The legend in the middle diagram reads, blue (vertical sides) = "parallel/congruent"; orange (horizontal sides) = "parallel/congruent"; pink (angles) = 90° angles.

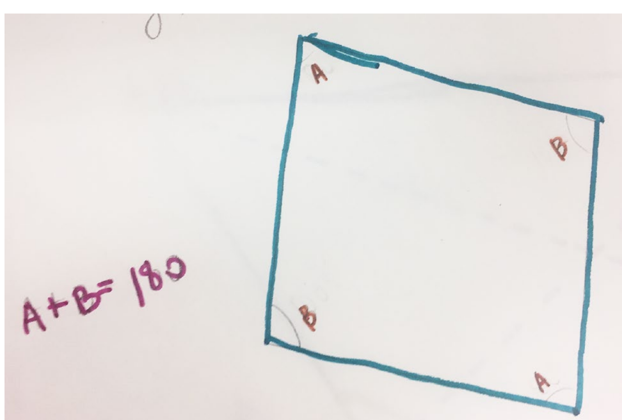
In addition to opportunities to construct diagrams and informally draw conclusions, the task also provided opportunities for students to make sense of each claim. This was especially true for the last three theorems in the task (Table 1), since the theorems referenced a different quadrilateral in the hypothesis and the conclusion. When constructing a diagram for the theorem, "if one angle of a parallelogram is a right angle, then the parallelogram is a rectangle", Arin, Sadie, and Brian initially constructed the figure by drawing a right angle and then a slanted line "because parallelograms have slant." As a result of only attending to the information in the hypotheses, their diagram resulted in a right trapezoid rather than a rectangle. Through a discussion with the TR, the students were able to connect their understanding of the definition of a parallelogram to recognize that they could construct a rectangle to satisfy both the theorem's hypotheses and conclusion. While the constructing diagrams task afforded more limited opportunities for reasoning-and-proving, the use of general claims for the task required students to make sense of the claims and afforded opportunities to informally draw conclusions from the hypotheses. Although often overlooked, reasoning about the claim itself can lay an important foundation to support students in the proof construction process (Cirillo & Herbst, 2011).

Attention to the Domains of the Claims in Relation to Diagrams

Since students were asked to use their own notation methods, this aspect of the diagrams provided insights into how they were attending to the domain of the claims. Of the nine diagrams students constructed in Session 7, four of them suggested that students paid limited attention to the domain of the claims either in the type of quadrilateral they drew, or in their selected notation method. In contrast, only one of the nine diagrams constructed in Session 8 contained specific notation that did not encompass all objects within its domain. For instance, Lauren, Megan, and Wilson first labeled the angles of their parallelogram 100º and 80º when constructing a diagram for the statement, "If a quadrilateral is a parallelogram, then the measures of the angles on the same side of the shape add to 180 degrees." When asked if those were the *only* angle measurements for a parallelogram, Wilson replied, "I don't know, those probably don't even, they add to 180 I know that, but those probably aren't the exact measurements you know." Even though this group originally labeled their angles with specific measurements, Wilson's justification suggests that he had chosen the angle measurements arbitrarily and had not based them on the actual measurements in their diagram. As a result of the TR's question, Megan proposed changing the labels to "A, A, B, B" and Wilson suggested adding the equation "A + B = 180" (Figure 3). Note that their small-group conversation did not reveal any evidence that they intended their diagram to be a rhombus instead of a parallelogram (Wilson: "just draw a parallelogram…just one that looks nice."). At the end of Session 7, both Megan and Wilson stated during the whole-class conversation that they preferred the use of variables to notate the sides of the rectangle because variables were "more generic."

Figure 3

Lauren, Megan, and Wilson's Revised Diagram for the Theorem, "If a Quadrilateral is a Parallelogram, then the Measures of the Angles on the Same Side of the Shape Add to 180 Degrees."

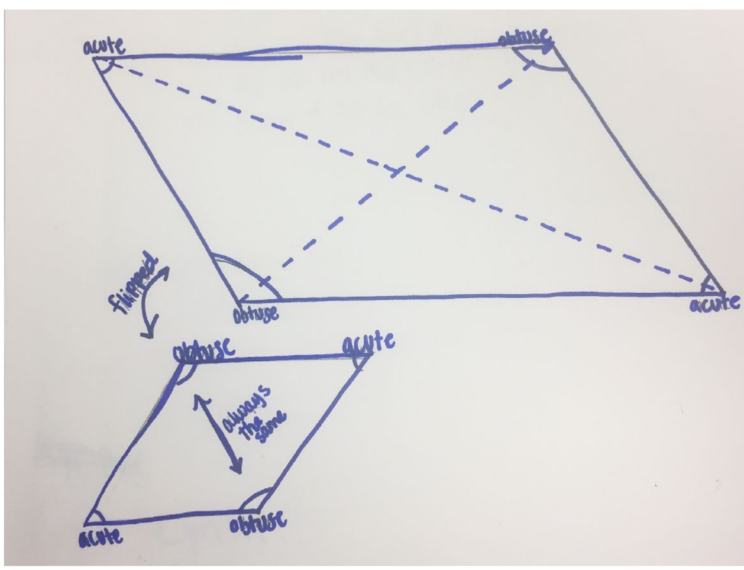


Arin, Brian, and Sadie's diagram for the same statement consisted of a general parallelogram; however, their decision to label the angles as "acute" and "obtuse", omitting right angles, made the notation less general than the prior group's use of variables (Figure 4). After being asked if the specific angles of parallelograms would always be acute and obtuse as they had labeled them, Arin replied "no, it could change. Like if the lines were drawn [in the opposite direction], then this [acute] angle would be obtuse." In response to this exchange, Sadie drew a second, smaller diagram containing different angle labels (Figure 4). Although the constructed parallelogram is generic, their choice to label the angles as obtuse or acute could result in a corresponding mathematical argument that makes assumptions about the angles that are not true for all cases (e.g., the upper left angle is acute). Sadie's decision highlights one of the challenges of constructing diagrams for general statements: namely, that it is impossible to construct a diagram that has the features of *all* possible shapes.

At the beginning of the constructing diagrams task, there was limited evidence that students were attending to the domain of the claims. However, their attention to the domain of the claim increased after the TR questioned groups whether their diagram applied to all possible shapes. At the end of Session 8, students appeared to have a greater awareness of the different ways that diagrams could be drawn to represent general claims. In a written reflection, Amanda explained that "it is okay that the diagrams didn't look the same because not all shapes may look the same, but they still fit the requirements to be that shape." Amanda and others recognized that diagrams can vary in their appearance so long as they contain all of the features specific to the shape mentioned in the general claim. Students' use of their own notation methods when labeling diagrams for general claims allowed for greater insight into their attention to the generality of the claims and highlighted the challenge students can face in constructing a single diagram to represent a class of objects.

Figure 4

Arin, Sadie, and Brian's Diagram for the Theorem, "If a Quadrilateral is a Parallelogram, then the Measures of the Angles on the Same Side of the Shape Add to 180 Degrees."



Proving Similar Polygon Conjectures Task

Varied Opportunities for Reasoning-and-Proving

The proving similar polygon conjectures task, as set up by the TR in Session 11, provided opportunities for students to pose conjectures about certain polygons that might be similar (e.g., "all squares are similar"), investigate the validity of their conjectures, either construct a proof or find a counterexample, construct a diagram to accompany their argument, evaluate their peers' arguments, and revise their argument based on peer feedback. In addition to engaging in all of the reasoning-and-proving activities set forth in the task, some students also constructed non-proof rationales and posed a revised conjecture while evaluating their peers' arguments.

To illustrate the different reasoning-and-proving opportunities embedded within this task, we describe the original argument constructed by Group 1 (Arin, Brian, and Sadie), the feedback given to Group 1 by Group 2 (Megan, Wilson, and Lauren) and Group 3 (Clay, Amanda, Heather, and Lexi), and then the revisions Group 1 made to their argument in response to the provided feedback (see Appendix B for final work). Group 1 worked on the claim that all squares are similar. Group 1's original argument included a diagram consisting of two different-sized squares with no notation on the sides and the angles labeled with the variables A, B, C, and D. Their written argument stated: "If all of the angles on a square are 90 degree angles, then they are the same. If all sides have the same measurements, then they will be proportional." Note that each sentence in their argument addressed one component of the definition for similar polygons. However, they did not justify how they knew the sides would be proportional or explicitly mention the definition of similar polygons or squares.

When evaluating the argument written by Group 1, Megan, Wilson, and Lauren focused on the way the group had chosen to label their diagram.

> *Megan*: I think that they should put…like, if they're going to do like letters then there should be ones on the sides too because that's, like, what makes it a square.
>
> *Wilson*: I think they should all be (unintelligible), I don't think they should be A, B, C, D, I think it should be A, A, A, A cause they're all the same angle.

Notice that both Megan and Wilson provided a non-proof rationale (e.g., "they're all the same angle") to justify their proposed revisions. After a discussion with the TR, they concluded that the angles could be labeled with 90° instead of a variable. This groups' final feedback included a revised diagram with the angles all labeled 90° along with the statement, "If the angles are the same, the side measurements will be proportional." Their feedback assumed a relationship between congruent angles and proportional sides, however it had not yet been discussed how to demonstrate that the sides of squares were proportional for all cases.

Clay, Amanda, Heather, and Lexi (Group 3) provided feedback by underlining Group 1's use of the words "same" and "proportional" at the end of each sentence and then writing, "We're not trying to prove that they are proportional, but that they are similar." This critique suggests that they may not have recognized that each sentence in the original argument referred to one of the components of the definition for similar polygons. Nonetheless, it highlighted the need for Group 1 to use more precise language in their original argument or to more clearly lay out the broad goals of their argument. During the revision process, Group 1 tweaked the first sentence to clarify that the *angles* are the same in response to Group 3's feedback. They also revised their angle notation in the diagram and added labels to the sides of the two squares in response to Group 2's feedback.

In the proving similar polygons task, students had the opportunity to engage in a variety of reasoning-and-proving activities as they developed and honed their understanding of proof. Although none of the groups produced arguments that contained all of the elements and formatting of a traditional proof (which, after all, was not expected), their work on the task was notable given that this was their first formal experience constructing a proof. Students were actively involved in the decision-making during this task, evaluated each other's feedback, and decided whether they wanted to incorporate it into their revised argument. The directions within this task not only allowed students to experience varied reasoning-and-proving in a connected, authentic way, but also allowed students to enact their role as a member of the proving community through revising their argument based on peer feedback.

Attending to the Domain of the Claim in Varied and Complex Ways

When proving that all squares are similar to one another, all three groups wrote arguments containing justifications that reflected an attention to the domain of the claim. Additionally, two of the three groups constructed a single diagram to accompany their argument that used variables to label the sides and, in one group, the angles as well. After completing their initial written argument, the remaining group (Heather, Amanda, Lexi, and Clay) chose to "draw another square", which they labeled with specific side lengths, "to show that they all work". It is not clear from this group's discussion whether they saw the specific diagrams as part of their core mathematical argument or as further evidence to convince someone the claim was true. Finally, all three groups also demonstrated understanding that a single counterexample proved a general claim was false for the right triangle conjecture.

Students' arguments for the (false) claim that all rhombuses are similar revealed more variation in attention to the domain of the claims, in part because they were completed individually during the final interview instead of within their small groups. Six of the ten students demonstrated understanding of the domain of the claim, "all rhombuses are similar", through their use of a single counterexample or class of counterexamples to prove the claim was false. Specifically, they argued the claim was false by giving a specific counterexample (Amanda, Lauren, and Sadie), mentioning that the angles of a rhombus "have no set rule" (Megan), or providing a class of counterexamples with squares and non-square rhombuses (Lexi and Wilson). For example, Wilson's written argument is shown below:

> By definition a rhombus is a polygon that has 4 equal sides that the angles add up to 360°. By definition a square has 4 90 degree angles with sides that are equal. A rhombus doesn't have to have 90° angles and a square does. Because of this 2 rhombuses don't necessarily have to be similar.

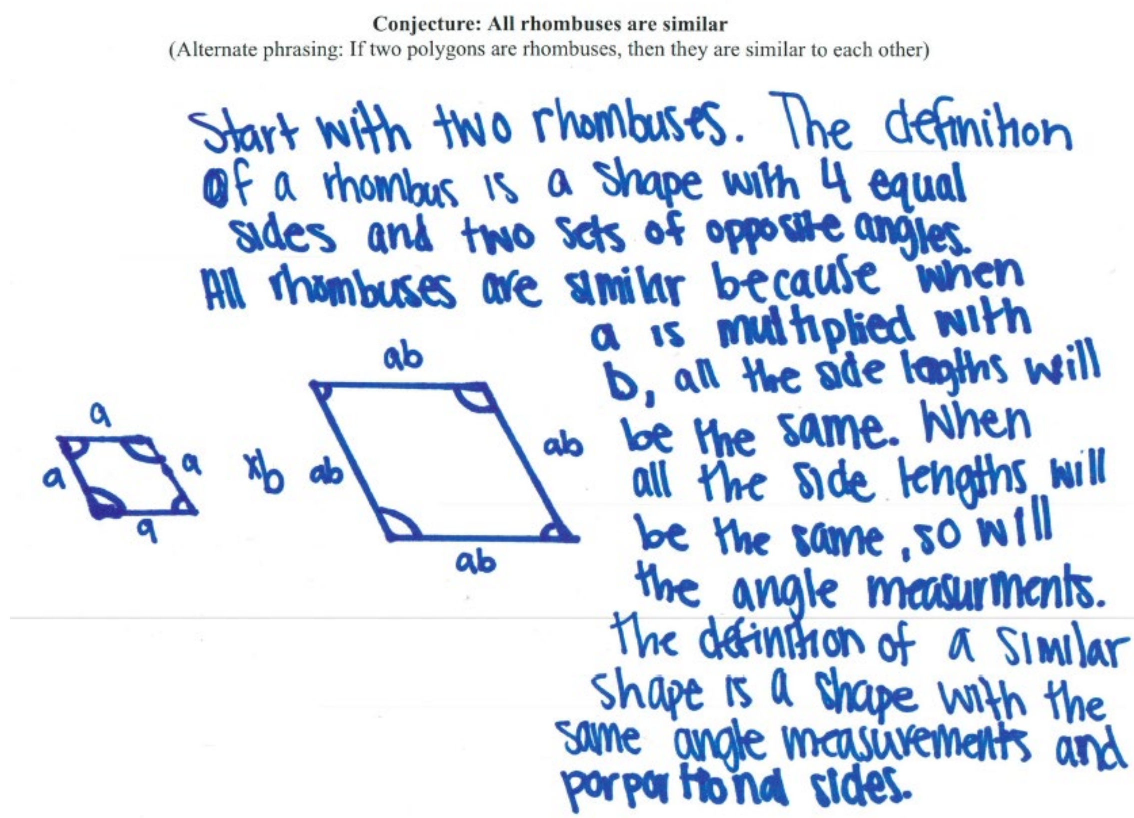
Wilson's argument demonstrates understanding that in order for the claim to be true, it must be true for all possible cases even though he did not disprove the claim with just a single counterexample. Whether by providing a single counterexample, or a class of counterexamples, to prove the claim was false, the six students demonstrated attention to the fact that the claim must be true for *all* rhombuses in order to be considered true.

The arguments produced by the remaining four students (Arin, Brian, Clay, and Heather), who initially thought the claim "all rhombuses are similar" was true, also demonstrated some attention to the domain of the claim. However, the way they conceived of the claim resulted in them considering only a subset of rhombuses. Arin, Brian, and Clay's arguments assumed that the angle measurements would stay the same as the sides proportionally changed, while Heather only mentioned proportional side lengths (not equal angle measures) when stating the definition of similar polygons. In order to illustrate how Arin, Brian, and Clay were thinking about the claim, we focus on Arin's argument, shown in Figure 5.

Arin appropriately defined a rhombus and stated the definition of similar polygons, but incorrectly claimed, "When all of the side lengths will be the same, so will the angle measurements." She verbally justified this claim saying, "if the shape's proportional, then it'll just… it'll like make the, um, the shapes more bigger, but the angle measurements will stay the same because the shape isn't changing its shape, it's just changing its size." This additional information suggests that she viewed one of the rhombuses as a dilation of the other. Instead of thinking about the conjecture as selecting two arbitrary rhombuses and then determining if they were similar, she appeared to be thinking about the task as selecting one arbitrary rhombus and then dilating it to create the second rhombus. When asked whether rhombuses have particular angle measurements, Arin stated that the opposite angles "have to be the same, but other than that, they don't have to be specific." This reply further confirms our interpretation that her belief in the claim's validity was based on her understanding of similarity and the domain of the conjecture. Of the four students who initially thought the claim was true, Arin, Brian, and Clay appeared to have at least a surface-level understanding of the definition of similar polygons (i.e., could state the definition), which suggests that their initial belief that the claim was true was not due to a lack of content knowledge. Instead, their initial assertion that rhombuses are all similar appeared to be rooted in how they were interpreting the domain of the mathematical claim, that is, how they brought to mind "all" rhombuses. Overall, students' work on the similar rhombuses proof tasks highlighted the abstract level of thinking needed to fully grasp what it means to prove that a general claim is always true and raises the question of how to support students in developing such understanding. We next discuss some of these points.

Figure 5

Arin's Written Argument for the Similar Rhombuses Proof Task



Discussion

By examining the enactment of general claim proof tasks with respect to students' opportunities to (1) engage in reasoning-and-proving activity and (2) consider the domain of the claims, this study extends Otten, Gilbertson et al.'s (2014) focus on the nature of mathematical statements found in reasoning-and-proving tasks in Geometry textbooks. With respect to the opportunities for reasoning-and-proving (RQ1), we found that the general proof tasks provided opportunities for students to actively engage in all of the intended reasoning-and-proving activities. Additionally, students also went beyond the intended activities by making conjectures/claims, posing counterexamples in response to a peer's claim, and refining a peer's conjecture during the tessellation tasks; drawing conclusions and making sense of the claims during the constructing diagrams task; and providing non-proof rationales and revising a conjecture during the proving similar polygons task. The tessellation task and proving similar polygons task in particular provided opportunities for students to engage in reasoning-and-proving in an integrated manner, mirroring the intent behind the hyphenated term (G. J. Stylianides, 2008). Across the three tasks, these students who were new to proof engaged in all of the reasoning-and-proving activity identified in Otten, Gilbertson et al.'s (2014) framework, including multiple opportunities to construct a proof. Consequently, these general proof tasks provided students with opportunities to develop their understanding of proof by engaging in the reasoning-and-proving process, something that Otten, Males, and Gilbertson (2014) noted was lacking within the introduction to proof chapters of many U.S. Geometry textbooks.

Although not a focus of the present study, opportunities for additional, unplanned reasoning-and-proving activity surfaced in part due to several factors, including the use of a launch-explore-summarize lesson structure (e.g., Lampert, 2001; Stylianou, 2010), which engendered opportunities for students to make sense of the tasks in small groups before being given more formal instruction. There was also a sense of at least partially shared authority, with the expectation that students consider and respond to their peers' ideas (e.g., the whole-class conversation in the tessellation task), rather than looking to the TR for validation. In the case of the tessellation tasks, the additional reasoning-and-proving occurred as students worked to make sense of two general claims ("do all quadrilaterals tessellate?" and "do all regular polygons tessellate?") that had parallel structure (i.e., both investigated whether a particular class of shapes would tessellate) but differed in validity. When considered together, the two claims motivated a need for a non-empirical justification that explained *why* only some polygons tessellate. The factors we have proposed that may have positively influenced students' opportunities for reasoning-and-proving align with the idea that opportunities to learn extend beyond the specific tasks given to students. Other factors have also been found to be important, such as "the emphasis teachers place on different learning goals and different topics, […], the kinds of questions they ask and the responses they accept, [and] the nature of the discussions they lead" (Hiebert & Grouws, 2007, p. 379). Given that prior classroom studies have documented instances where teachers began by modeling the proof construction process and retained most of the mathematical authority (e.g., Harel & Rabin, 2010; Martin & McCrone, 2003; Otten et al., 2017), future studies could analyze specific instructional features that facilitate opportunities for students to engage in reasoning-and-proving that extends beyond the opportunities within the original task.

Analysis of students' attention to the domain of the claims during the three tasks (RQ2) highlighted the complexity of addressing generality and the different ways it impacts the reasoning-and-proving process. Specifically, attempting to consider all possible cases when investigating the validity of the claim (tessellation tasks) required a different shift in attention than depicting the generality of a claim when constructing and notating a diagram (constructing diagrams task) or proving a claim to be true for all possible cases (proving similar polygons task). In all three cases, it seemed to be important that the claims themselves were general, as opposed to an introduction-to-proof unit that presents simple, particular proofs (e.g., "write down the justifications for how we know that this segment of the given diagram is congruent to this other segment"). Although the general proof tasks afforded certain opportunities, as discussed above, they worked in concert with other factors such as the TR questions and the interactive dynamics. Moreover, the transfer of any attention to generality is not guaranteed, evidenced by the varied attention to generality during the similar rhombus task (final interview) despite everyone attending to generality during the similar squares task (Session 11).

We also wish to comment on the attention to the domain of claims over time. Towards the end of the study, students' work began reflecting an increased attention to the domain of the claims. We viewed this as a positive development as this was students' first formal introduction to proving. Yet the attention to the domain near the end of the study was nuanced. Whereas all student work demonstrated at least some attention to the domain of the claims in the proving similar polygons task, some students interpreted the claims in a way that only encompassed a subset of cases (e.g., Arin's argument in Figure 5). In the case of Arin, subsequent conversation suggested that her initial (incorrect) belief that all rhombuses were similar was not a result of a lack of content knowledge, but rather how she was thinking about the claim itself. Although Arin's work does not discount the role that content knowledge and proof skills play in understanding why some students construct empirical arguments for general claims, it does reinforce the particular difficulties students face in interpreting diagrams as figural concepts (Fischbein, 1993) and the need to better understand how students interpret the domain of mathematical claims (Mason, 2019).

Given this study's small sample size involving accelerated students and the explicit emphasis placed on the domain of the claim by the TR, more research is needed to ascertain the extent to which a wider range of students recognize the domain of the claim while engaging in reasoning-and-proving tasks. It is possible that students' prior successes in mathematics and their involvement in the accelerated mathematics program may be a form of selection bias contributing to the findings of additional, unplanned forms of reasoning-and-proving. That said, the accelerated program had only focused on algebraic topics at the time of the study, so the students' content knowledge was likely not significantly different from other students at the school as they began studying secondary geometry. Future research should involve students who were taught using a more traditional curriculum where particular claims are frequently used (Otten, Gilbertson et al., 2014) and the domain of the claim is often obscured through the use of separate "given" and "to prove" statements (Chazan, 1993).

How should we view students' work on the three tasks, given that they were completed as they were first being introduced to proof? Viewing students' work in the teaching experiment solely through the lens of the enacted opportunities to learn (RQ 1) paints a rosy but incomplete picture of their developed understanding of proof and ability to construct deductive arguments. On the one hand, there was evidence of careful attention to the generality of claims in nearly all of students' constructed arguments during the proving similar polygon conjectures task and on proof tasks given in the final interview (Conner, K.A., 2018). On the other hand, findings on students' attention to the domain of the claims (RQ 2) portrayed a more nuanced picture, in which students demonstrated attention to the domain of the claims in some instances, but not in others. In both the constructing diagrams task and proving similar polygons conjectures task, at least one group used specific numbers when labeling aspects of their diagram that can vary, even in instances when they explicitly referenced the domain of the claim in their language. One way to interpret students' inconsistent shifts in attention is to conclude that they developed understanding of generality at a surface level (i.e., they recognized the types of justifications that were appropriate and the need to prove the claim for all possible cases), but had not fully become aware of other aspects of proof that are impacted by it. Nevertheless, for an introductory unit, the fact that general proof tasks helped set the stage for a focus on the domain of claims and students began to discuss those domains, even if imperfectly, it may be a sufficient foundation on which to build. As the field continues to explore ways to improve the teaching and learning of proof, more research is needed on ways to support shifts in attention with regard to the level of generality indicated within a mathematical claim, and its impact in the reasoning-and-proving process.

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Appendix A

Overview of the Instructional Sequence

Session	Classroom Activities	Rationale for Tasks
1	"Do all quadrilaterals tessellate?"	Allowed for testing of specific cases where the cases seem unique due to differences in the diagrams. The task proof allowed for explanations of *why* it was always true.
2	Create step-by-step directions that explain how to tessellate *any* quadrilateral.	Aimed to facilitate systematic work and the identification of cross-cutting features of quadrilaterals that result in the figure tessellating.
3	"Do all regular polygons tessellate?" "Do you still think that all quadrilaterals tessellate? If no, explain why. If yes, is there something special about quadrilaterals that make it so that they will always tessellate?"	First question served as a pivotal counterexample (Stylianides & Stylianides, 2009) to cast doubt on their prior confidence that all quadrilaterals tessellate based on checking specific cases. The second question emphasized that a claim must be true for *all* cases and motivated the determination *why* a polygon will or will not tessellate.
4	Summary of first three sessions; TR explained why all quadrilaterals, but not all regular polygons, tessellate.	Introduced the explanatory feature of proofs.
5	Circle and Spots problem and Monstrous Counterexample (Stylianides & Stylianides, 2009)	Aimed to cast doubt on the idea of using examples to determine whether a statement is *always* true.
6	Introduce generic examples through exploration of a number trick: https://nrich.maths.org/2280. Next, students explored and proved: "9*11 equals 1 less than 10^2, 3*5 equals 1 less than 4^2. Will this pattern always be the case?"	Aimed to support students in interpreting and using geometric diagrams where they only attended to the features that extended across all cases within the domain. The second task aimed to facilitate interpretation and use of variables as varying quantities.
7 8	Students constructed diagrams for six quadrilateral theorems	Introduced conditional statements, notation methods, and what can/cannot be assumed true based on a geometric diagram.
9 10	Develop definition of similar polygons; based on sequence in Kobiela and Lehrer (2015)	Established necessary mathematical content knowledge for Sessions 11 and 12.
11	Students posed conjectures of the form "all ___ are similar", drafted an argument for squares, critiqued peer arguments, revised their	Developed understanding of proof by engaging in multiple aspects of the reasoning-and-proving process in small groups. The

12	arguments, then discussed proof as a whole class. Next, students investigated remaining claims from previous sessions.	second session was used to introduce specific characteristics of proofs.
13	Students individually engaged in the reasoning-and-proving process (described in sessions 11-12) for the exterior angle theorem. Task was posed using two examples, followed by the question "is this a coincidence?"	Developed understanding of proof by engaging in the reasoning-and-proving process. Individual written work was used as a formative assessment.
14	Students developed shared criteria for features of "good proofs"; task based on Boyle and colleagues (2015).	Assessed conceptions of proofs and reflected on key ideas from the teaching experiment.

Appendix B

Brian, Arin, and Sadie's (Group 1) revised argument for the conjecture, "all squares are similar", including the feedback given by Group 2 (bottom) and Group 3 (top right).