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Mathematical Representations in the Teaching and Learning of Geometry: A Review of the Literature from the United States

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ABSTRACT

This paper presents a synthesis of the literature exploring the teaching and learning of geometry and the role that mathematical representations can play in enriching geometry experiences for our students. Geometry is the only content domain to be taught in all PK-12 grades, however, from historical trends in international assessment data, it continues to be a low scoring area for students in the United States. This paper is organized by the following: (1) theories guiding the teaching and learning of geometry in the U.S.; (2) teaching and learning of geometry in the U.S.; and (3) the role of mathematical representations in geometry. In order for students to appreciate and experience the wonder, joy, and beauty of geometry in a consistent and coherent manner, they need geometry learning experiences that leverage high quality tasks with opportunities in translating *between* and *within* multiple representations, and engage them in discovering connections within geometry, between geometry and the other mathematics content domains, and between geometry and their world.

Keywords: geometry, high quality tasks, mathematical representations, mathematics education

Introduction

In the 2001 National Council of Teachers of Mathematics (NCTM) Yearbook, Albert Cuoco challenged the mathematics education community to think beyond ideas of content and pedagogy, to how our students learn (NCTM, 2001). Most recently, one way NCTM has addressed this call is through the introduction of the eight mathematics teaching practices (NCTM, 2014). The aforementioned practices provide students the opportunity to access mathematics through multiple entry points while leveraging multiple mathematical representations (i.e. visual, symbolic, verbal, contextual, and physical) (Lesh et al., 1987). During the past decades, mathematical representations have been defined in multiple ways (Goldin, 2014; Huinker, 2015; NCTM, 2014). For the purpose of this paper, mathematical representations will refer to the five types that were initially defined by Lesh and colleagues (1987), which we describe in more depth later in this paper. Huinker and Bill (2017) refer to the importance of students using multiple mathematical representations - both *between* representation types and *within* the same representation type. The ways in which these connections among mathematical representations can be leveraged - specifically in geometry instruction - will also be discussed in this paper.

Some effective uses of mathematical representations include connecting instruction with students' experiences and interests (NCTM, 2018). Teaching geometry is crucial in facilitating student opportunities to make connections with the real world (Usiskin, 1980), in addition to experiencing geometry in an integrated and active manner capitalizing on the wonder, joy and beauty of examining the world (NCTM, 2020a). The study of geometry and measurement provides rich opportunities for children to both explore and visualize the two- and three-dimensional, and represent objects and the relationships between them, and enrich and connect geometrical ideas to other mathematical domains and the world around them (NCTM 2020a, 2020b).

Gonzáles and Herbst (2006) identify four aims for the teaching of geometry: (a) a *formal* argument: geometry teaches to use logical reasoning; (b) a *utilitarian* argument: geometry serves to prepare students for the workplace; (c) a *mathematical* argument: geometry for the experience and the ideas of mathematicians; and (d) an *intuitive* argument: geometric expression helps students interpret their experiences in the world. Prior research (i.e. International Commission on Mathematical Instruction, ICMI, 1995) has shown that there is no linear, hierarchical path from beginning to more advanced geometry – geometric ideas must be examined, reconsidered, reimagined, and refined at different stages from different viewpoints.

Among mathematicians and mathematics educators, there is widespread agreement that teaching geometry should start at an early age and should continue throughout the entire mathematics curriculum (ICMI, 1995). This is also illustrated in the Common Core State Standards for Mathematics (2010) progression (and other similar college and career readiness standards) which list geometry as the only domain taught in all PK-12 grades. This makes it evident that current reform-based curriculum supports revisiting geometric ideas, however, historically speaking, "Geometry has been treated solely as geometry and not as a subject, which in addition to being a splendid example of deductive reasoning, important and interesting in itself, can also serve the purpose of creating a critical attitude of mind toward deduction and thinking in general" (NCTM, 1940, p. 39). In many countries, geometry has also lost its former central position in mathematics teaching – the subject is often somewhat ignored or confined to the teaching of facts about figures and their properties (ICMI, 1995). In the U.S., students revisit the subject every year, yet, they are often given too little exposure to geometrical thinking in grades K–8, particularly in the middle grades, so their understanding of geometry does not always develop to deeper levels of analysis (Clements, 2003; Clements & Battista, 1992; Driscoll, 2007; Steele, 2013). Several researchers have supported the idea that an increased focus on researching and understanding the place of geometry in curriculum would be well advised (Fuys et al., 1988; Sinclair & Bruce, 2015).

While ideas about the use of mathematical representations have been researched, as well as about geometry curriculum, there is little literature that synthesizes both. Individual representations cannot fully describe a mathematical construct, and each has different advantages. Therefore it becomes crucial that we expose students to using multiple mathematical representations. This allows students to appropriately choose the representation(s) that best works for the given context (Duval, 2002) and for themselves as learners. This literature review provides a synthesis on the teaching and learning of geometry at the PK-12 level and the role that mathematical representations can play in enriching the geometry experience for our students.

The following research questions guided this review of literature:

(1) *Which frameworks have guided the teaching and learning of geometry in the U.S.?*

(2) *What does the teaching and learning of geometry in the U.S. look like?*

(3) *What is the role of mathematical representations in the teaching and learning of geometry?*

Literature Search Procedures

In order to conduct a thorough review of scholarly literature, an organized search process was used drawing from a variety of databases including EBSCO Academic Search Premier, Education Resources Information Center (ERIC), Science Direct, and JSTOR. The search terms used include "geometry" AND "representations", "geometry curriculum", "geometry curriculum" AND "representations", and "mathematical representations". This initial search resulted in more than 200 pieces of literature which was then narrowed to include only those written only in English. While the authors acknowledge the influence of international perspectives on the teaching and learning of geometry in the U.S., for the purpose of this literature review, any papers discussing this topic in non-U.S. settings were excluded in order to maintain the focus on U.S. PK-12 education. Additionally, any not relating to the previously stated operational definition of mathematical representations for this paper (e.g., articles relating to racial or cultural representations in mathematics) were also excluded as they were deemed beyond the scope of this study. After conducting this search, additional sources were found using the reference lists of each included article. In all, sixty-seven items of scholarly literature consisting of peer-reviewed journal articles and books were included in this paper. The results of this literature search determined the structure of this paper which is organized by the following: (1) frameworks guiding the teaching and learning of geometry in the U.S.; (2) teaching and learning of geometry in the U.S.; and (3) the role of mathematical representations in the teaching and learning of geometry in the U.S..

Frameworks Guiding the Teaching and Learning of Geometry in the U.S.

While the focus of this literature review is to discuss the role of mathematical representations in the teaching and learning of geometry, it is necessary to first consider the frameworks that have influenced geometry instruction. For purposes of this paper, we define frameworks broadly as contributions that are theoretical frameworks, conceptual frameworks, conceptual models, theories, or similar. This section addresses research question one through a discussion of eight frameworks which provide the theoretical background to guide and support research on the teaching and learning of geometry. These frameworks found through the literature search are foundational in understanding how geometry teaching and learning has evolved over time. A summary is provided in Table 1, and details for each framework is provided in this section.

First, Van Hiele's (1986) framework postulates that the five levels of geometric thinking were sequential and hierarchical, and that for students to attain the next level, they must pass through the preceding one. These five levels are visual, analytic, abstract, deductive, and rigor which describe children's levels of thought in learning geometry. While some previous research suggested that these levels accurately describe the development of students' geometric thinking (Burger & Shaughnessy, 1986; Clements & Battista, 1992), in recent decades, researchers are beginning to argue that students may develop their thinking in these various levels simultaneously (Battista, 2007; Clements, 1999). These later researchers maintain that the third framework, abstraction theory (Battista & Clements, 1996), which proposes that learning is a recursive cycle through phases of action, reflection, and abstraction - may reflect a more accurate way to describe students' geometric thinking.

The second framework– the theory of figural concepts (Fischbein, 1993) - attempted to interpret geometrical figures as mental entities that simultaneously possess conceptual and figural properties. According to this notion of figural concepts, Jones (1998) describes that geometrical reasoning is characterized by the interaction between the figural and the conceptual aspects. It is necessary for students to form connections between both the conceptual (abstract) representation and the visual representation, however that is generally where students make the most errors (Jones, 1998).

Table 1

Frameworks for Teaching and Learning Geometry

Name	Description
van Hiele (1986)	Children move through five levels of thought in geometry - visual, analytic, abstract, deductive, and rigor.
Theory of Figural Concepts (Fischbein, 1993)	Geometric figures are mental entities which simultaneously possess conceptual and figural properties.
Abstraction (Battista & Clements, 1996)	The process by which the mind registers objects, actions, and ideas in consciousness and memory, and further describes two forms – spatial structuring, and mental models.
Cognitive Processes (Duval, 1998)	Geometrical reasoning involves three kinds of cognitive processes which fulfill specific epistemological functions: visualization processes, construction processes, reasoning processes.
Geometric habits of mind (Driscoll, 2007)	Teachers need to develop an understanding of geometric thinking and their own geometric habit of mind including: Reasoning with relationships, generalizing geometric ideas, investigating invariants, balancing exploration and reflection.
Concept learning and the objects of geometric analysis (Battista, 2009)	Students need to analyze objects (physical objects, concepts, and concepts definition) and mental entities to understand and reason about mathematics.
Diagrams and representations (Battista, 2009)	Both diagrams, and physical objects play a major role in geometry.
Spaces for geometric work (SGW) (Goméz-Chacón & Kuzniak, 2015)	Describes the work that people (students, teachers, mathematicians, etc.) perform when they solve geometric tasks.

Duval (1998), illustrating the fourth framework, approached geometric reasoning from a cognitive and perceptual lens. He described three cognitive processes which fulfill specific epistemological functions: (1) visual processes which refer to the visual representation of a geometrical statement or the heuristic exploration of a complex geometrical situation; (2) construction processes which refer to the use of various tools; and (3) reasoning processes which refers to the discursive processes for the extension of knowledge, for explanations, and for proofs. He further stated that these processes can be performed separately. In fact, he suggested that these three processes should be developed separately, and that it is necessary to differentiate between them before using them in coordination with one another.

In the fifth framework, Driscoll (2007) shares that teachers need to foster geometric thinking in their classrooms so that students will learn to use geometric thinking as a complement to algebraic thinking in problem solving. He describes that people with mathematical power perform thought experiments, invent things, look for invariants or patterns, make reasonable conjectures, describe things both casually and formally, think about methods, strategies, and processes, visualize things, and seek to explain why things are as they see them. To accomplish this goal, he proposes four geometric habits of mind that teachers need to develop which are: reasoning with relationships, generalizing geometric ideas, investigating invariants, and balancing exploration and reflection. These habits of mind allow teachers productive ways of thinking that enable them to support their students in learning and application of formal mathematics.

The sixth and seventh frameworks, by Battista (2009), further discuss the need for forming concepts from physical objects. He described "geometry instruction and curricula generally neglect the process of forming concepts from physical objects and instead focus on using diagrams and objects to represent formal shape concepts" (p. 97). Often, instruction moves too quickly away from physical manipulatives to diagrams and abstract thinking, or teachers avoid using manipulatives all together, and as a result students often incorrectly connect attributes of a diagram or object to the geometric concept. Students experience a world filled with physical objects, and in order to provide them opportunities to connect mathematics to their world, these physical objects play a crucial role.

In the eighth framework, spaces for geometric work (SGW) explained by Goméz-Chacón & Kuzniak (2015), describes the process that is performed when thinking about geometric tasks. SGW describes two interconnected planes: the epistemological and the cognitive (Kuzniak, 2015). The epistemological plane contains three intersecting elements: (a) real and local space as material support with a set of concrete and tangible objects; (b) artifacts such as drawing instruments or software; and (c) a theoretical frame of reference based on geometric definitions and properties. The cognitive plane (adapted from Duval, 1998) is comprised of three cognitive processes: (a) visualization process connected to the representation of space and material support; (b) construction process determined by instruments (ruler, compass, etc.); and (c) a discursive process which conveys argumentation and proofs. Both the epistemological and cognitive planes are interconnected through the synthesis between three different modes of knowledge: intuition, experiment, and deduction and both need to be articulated in order to ensure complete geometric work (Houdement & Kuzniak, 2003). Although the SGW model was developed for geometry it can also be generalized and connected to other mathematical domains.

In summary, these eight frameworks represent some of the long-standing and current frameworks on geometry teaching and learning. These frameworks lay the necessary foundation for further understanding research on the teaching and learning of geometry in the U.S.. Mathematics is filled with connections between and within domains and the use of mathematical representations allow us to make and leverage these connections. The world students live in is full of shapes with some exhibiting beautiful consistent patterns while others seem to lack symmetry or regularity. Opportunities that allow students to experience the "harmony, beauty, order, clarity, wonder, curiosity, and enjoyment of mathematics" (NCTM, 2020b, p. 15) are important in their development of a positive mathematical identity. While simple formulas are used in school mathematics, they do not account for the irregularity. Therefore, it is crucial for our students to be exposed to the messiness in mathematics that exists in the world around them, and having a toolbox of multiple mathematical representations allows them to make sense of this (Organisation for Economic Co-operation and Development, OECD, 2018). This idea of making important and necessary connections using various types of mathematical representations (i.e. Lesh et. al, 1987; Huinker & Bill, 2017) - visual, symbolic, verbal, contextual, and physical) during geometry instruction will be explored further in a later section.

Teaching and Learning of Geometry in the U.S.

This section addresses research question two to understand why representations as well as connections among representations are essential in PK-12 mathematics classrooms, specifically focusing on both the traditional and current approaches to teaching and learning of geometry. Geometry is one of the oldest branches of mathematics, and its origins can be traced back to a wide range of cultures and civilizations. Yet, the aims and goals of modern geometry instruction are widely debated (Jones, 2000; The Chicago School Mathematics Project, 1971). Jones (2000) states "The fundamental problem in the design of the geometry component of the mathematics curriculum is simply that there is too much interesting geometry, more than can be reasonably included in the mathematics curriculum" (p. 75). At least in North America, in over the past hundred years, high school geometry was comprised of students using Euclid's *Elements* (Sinclair, 2008). In the 1960s, geometry was then explicitly introduced as a topic in primary schools, and focused primarily on the study of two-dimensional geometry to prepare students for Euclidean geometry (ICMI, 1995).

More recent studies claim similar purposes for learning geometry and further extend the purpose of elementary school geometry to focus on spatial reasoning (Clements & Battista, 1992; Battista, 2007), and secondary geometry instruction to focus on dynamic geometry software (Hollebrands, 2003) and connections between geometry to algebraic and symbolic manipulations (Knuth, 2000). Geometry serves as an essential foundation for space and shape, and also draws on elements of other mathematical ideas such as spatial visualization, measurement and algebra (OECD, 2018). "[Geometry and measurement] are among the first mathematical ideas to emerge for young children as they interact with their environment and they deepen through early childhood and elementary mathematics" (NCTM, 2020a, p. 115). Table 2a and 2b include an overview of current geometry standards in the U.S. as denoted by the Common Core State Standards for Mathematics (CCSSO & NGA, 2010), but are similar for many states that have adopted college and career readiness standards. While Tables 2a and 2b show how geometry standards progress across the grade levels in the standards, it is important to consider that geometry is also connected to many other mathematical content domains, and this learning trajectory is a combination of developmental progression and an instructional sequence (as described in Clements & Sarama, 2004) . Mathematics standards are not isolated concepts – they are connected to each other both within and across grade levels. It is crucial for educators to understand these connections so they can link to students' prior knowledge while building a strong foundation for the connections that are still to come (Achieve the Core, n.d.).

Researchers have found that while students in the U.S. are given plenty of exposure to geometry, there is a lack of exposure to deep geometrical thinking, and that many teachers need further development to effectively teach it with depth (Clements, 2003; Driscoll, 2007; Steele, 2013). In fact, this is true among all mathematics domains where a majority of mathematics teachers report that instructional materials given to them provide opportunities to teach major topics addressed by state standards. However, a significantly lower percentage of teachers indicated that their materials addressed these topics with equal time, rigor, and intensity (Opfer et al., 2016). Looking at both *what* and *how* geometry is taught, it becomes evident that most U.S. geometry curricula tends to be scattered and while various topics are taught, much is explored at the surface level and does not support higher levels of geometric thinking (Clements & Battista, 1992; Senk, 1989; Sinclair & Bruce, 2015). This lack of instruction and exposure to deep geometrical thinking is also evidenced by U.S. students' performance on the international level, as evidenced by the TIMSS 2007, 2011, 2015, and 2019 data, which show that geometry has historically been the content domain with the lowest performance, and this is true across all tested grade levels (4^{th} grade, 8^{th} grade, end of high school) (Mullis et al., 2020).

Table 2a

Common Core State Standards - Geometry Standards (K-8)

Grade	Main Ideas
K	- Identify and describe shapes - Analyze, compare, create, and compose shapes
1	- Reason with shapes and their attributes
2	- Reason with shapes and their attributes
3	- Reason with shapes and their attributes
4	- Draw and identify lines and angles, and classify shapes by properties of their lines and angles
5	- Graph points on the coordinate plane to solve real-world and mathematical problems - Classify two-dimensional figures into categories based on their properties
6	- Solve real-world and mathematical problems involving area, surface area, and volume
7	- Draw, construct, and describe geometrical figures and describe the relationships between them - Solve real-life and mathematical problems involving measurement, area, surface area, and volume
8	- Understand congruence and similarity using physical models, transparencies, or geometry software - Understand and apply the Pythagorean theorem - Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres

Adapted from CCSSO and NGA (2010)

Table 2b

Common Core State Standards - Geometry Standards (High School)

Grade	Categories within Geometry	Main Ideas
High School	Congruence	- Experiment with transformations in the plane - Understand congruence in terms of rigid motions - Prove geometric theorems - Make geometric constructions
	Similarity, Right Triangles, and Trigonometry	- Understand similarity in terms of similarity transformations - Prove theorems involving similarity - Define trigonometric ratios and solve problems involving right triangles - Apply trigonometry to general triangles
	Circles	- Understand and apply theorems about circles - Find arc length and areas of sectors of circles
	Expressing Geometric Properties with Equations	- Translate between the geometric description and the equation for a conic section - Use coordinates to prove simple geometric theorems algebraically
	Geometric Measurement and Dimension	- Explain volume formulas and use them to solve problems - Visualize relationships between two-dimensional and three-dimensional objects
	Modeling with Geometry	- Apply geometric concepts in modeling situations

Adapted from CCSSO and NGA (2010)

The Role of Representations in Geometry

Bossé and Adu-Gyamfi (2011) describe six modalities of student learning in geometry – communication, collaboration, reading and writing, real-world examples, multiple representations, and technology. This section addresses research question three in describing the role of multiple mathematical representations in geometry instruction. NCTM (2000) recommends providing students opportunities to select, apply, and transfer among mathematical representations to solve problems. NCTM (2014) describes that one facet of effective teaching is to engage students in making connections among mathematical representations to deepen their mathematical understanding. All students arrive to class with prior formal and informal mathematical experiences, and using multiple mathematical representations allows students to draw on multiple sources of knowledge (Boston et al., 2017). By selecting tasks which allow for students to use multiple mathematical representations, teachers can value and encourage students to draw on their mathematical, social and cultural competence, thereby positioning students as being mathematically competent (Boston et al., 2017; Smith et al., 2017).

Lesh and colleagues (1987) proposed five different types of mathematical representations (i.e. visual, symbolic, verbal, contextual, and physical) which are relevant across mathematical content domains and the importance of making connections between them to deepen students' mathematical understanding. "[Students] will need to be able to convert flexibly among these representations. Much of the power of mathematics comes from being able to view and operate on objects from different perspectives" (NCTM, 2000, p. 361). In 2015, Huinker suggested a consideration to Lesh and colleagues (1987) mathematical representations classification by suggesting that there are two important types of translations that need to be developed: (a) translations *between* these different modes of representations such as translations from a visual model to an equation (adapted from Lesh et al., 1987; NCTM, 2014); and (b) translations *within* a specific mode of representation such as from one visual model to another visual model (e.g. comparing an array and an area model). While research supports the usefulness of representations and the rich mathematical perspectives that representations provide, transferring between and within representations can be challenging for both teachers and students alike. Teachers must be deliberate in creating experiences where students are given the opportunity to make sense of mathematical relationships using multiple mathematical representations (Boston et al., 2017). As students are expected to be flexible translating between and within mathematical representations, it is important for teachers to emphasize this as a part of their daily instruction, in turn influencing students' knowledge and ability to use various representations fluently.

In order to understand the role of mathematical representations that are present throughout the geometry curriculum it is first important to understand the progression of the main ideas in geometry (Figure 1) as shown in the Essential Understanding Geometry series (Dougherty et al., 2014; Goldenberg et al., 2014; Sinclair et al., 2012a, 2012b). Table 3 delves deeper into the big ideas in geometry at the K-2, 3-5, 6-8, and 9-12 level. These big ideas connect to the Common Core State Standards (CCSSO & NGA, 2010) described previously in Table 2a and 2b, and are further examined from a representational standpoint in the grade band subsections that follow.

Figure 1

Progression of Main Ideas in Geometry Based on Work of Dougherty et al., (2014); Goldenberg et al., (2014); Sinclair et al., (2012a), (2012b)

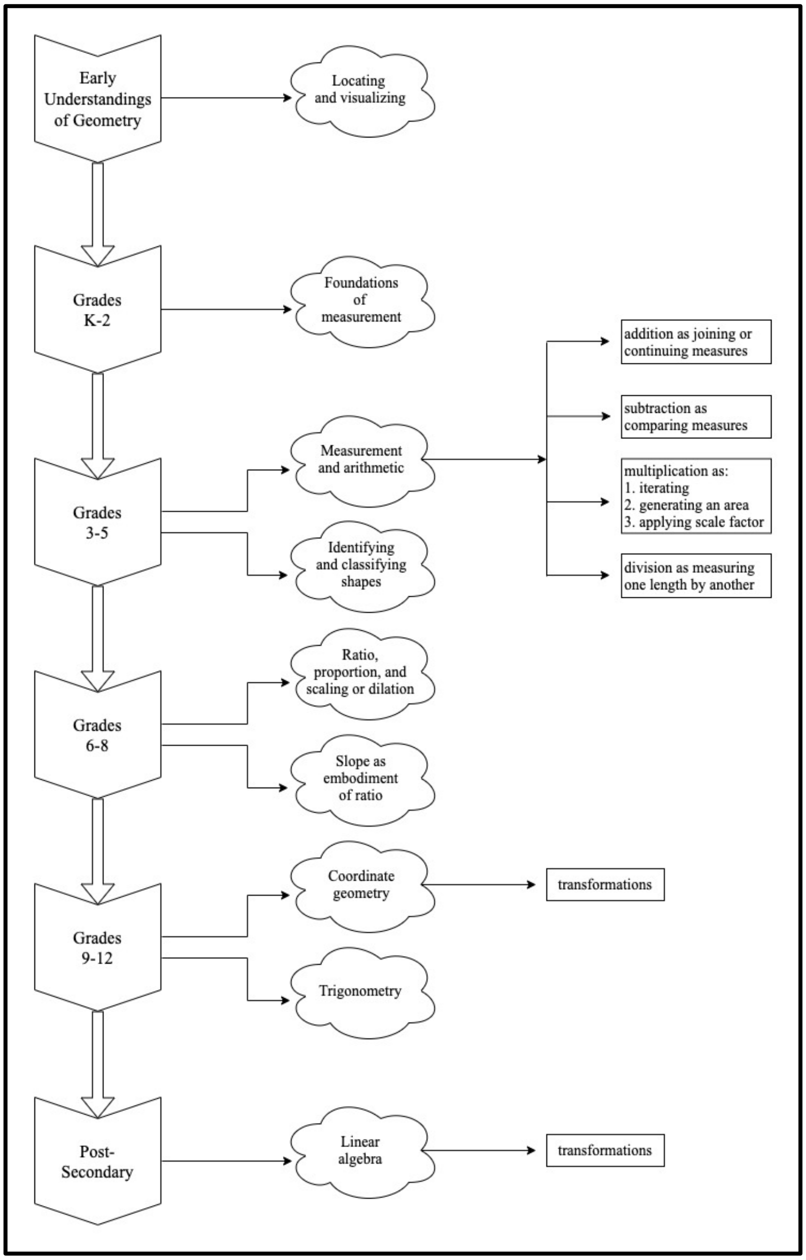

Table 3

Big Ideas in Geometry

Grade Band	Big Ideas
K-2	1: Classification scheme specifies for a space or the objects within it the properties that are relevant to particular goals and intentions. 2: Geometry allows us to structure spaces and specify locations within them. 3: We gain insight and understanding of spaces and the objects within them by noting what does and does not change as we transform these spaces and objects in various ways. 4: One way to analyze and describe geometric objects, relationships among them, or the spaces that they occupy is to quantify – measure or count – one or more of their attributes.
3-5	1: Transforming objects and the space that they occupy in various ways while noting what does and does not change provides insight into and understanding the objects and space. 2: One way to analyze and describe geometric objects, relationships among them, or the space that they occupy is to quantify – measure or count – one or more of their attributes. 3: A classification scheme specifies the properties of objects that are relevant to particular goals and intentions.
6-8	1: Behind every measurement formula lies a geometric result. 2: Geometric thinking involves developing, attending to, and learning how to work with imagery. 3: A geometric object is a mental object that, when constructed, carries with it traces of the tool or tools by which it was constructed. 4: Classifying, naming, defining, posing, conjecturing, and justifying are codependent activities in geometric investigation.
9-12	1: Working with diagrams is central to geometric thinking. 2: Geometry is about working with variance and invariance, despite appearing to be about theorems. 3: Working with and on definitions is central to geometry. 4: A written proof is the endpoint of the process of proving.

Compiled from Dougherty et al., (2014); Goldenberg et al., (2014); Sinclair et al., (2012a), (2012b)

Early Childhood and Elementary Geometry Experiences

From early childhood, the domains of geometry and spatial reasoning are an important area of mathematics learning. Geometry, just as with other areas of mathematics, is an extension of what we do naturally (Goldenberg et al., 2014). Without yet formalizing it, young children are able to understand the distance between themselves and their toys, change location and orientation, and can grasp edges and crawl and run around shapes. In a study involving pre-school participants, Villarroel and Ortega (2017) found that children naturally use geometric shapes in their art even before they have any formal experiences. These early understandings of geometry are supported in the literature (Dougherty et al., 2014; Goldenberg et al., 2014; Sinclair et al. 2012a, 2012b) which indicate that locating and visualizing are students' first introduction to geometry. All these initial exposures to geometric representations engage students in informal reasoning, which support and build a foundation for informal and formal reasoning in K-12 mathematics, and serve as a core in relating other subject areas to mathematics (Clements & Sarama, 2011).

Once students formally start school, students in grades K-2 start to spend time exploring geometry within the context of their own environments and then learn to start engaging in formal activities by identifying and describing the shapes they see and touch (Dixon et al., 2016). In grades 3-

5, students build a foundation of geometric ideas such as dividing shapes into equal pieces which connects to ideas even in high school, such as to trigonometric ratios such as sine, cosine, and tangent (Dixon et al., 2016). By initially forming connections between the visual, physical, and contextual representations, students are then able to develop formal language to describe the shapes. This progression of geometric understanding that students develop at the K-5 level is important to students' overall mathematical learning.

A study by Cai and Lester (2005) in U.S. and Chinese elementary schools found that the types of representations that students use heavily relies on the representations used by their teachers, thereby emphasizing the importance of using multiple mathematical representations during instruction. During a task implementation, Bay-Williams and Fletcher (2017) established that modifying the hundred charts to create an alternative bottom up representation better aligned the concrete and physical manipulatives with the language connected to children's geometrical thinking. Such representations allow for connections to representations that students are exposed to at the K-2 level and beyond, such as physically stacking objects, counting using number lines, and extending to graphing on a coordinate axis. The use of such representations is also supported by Huinker and Bill (2017) who suggest that in these grade levels, visual and physical representations are particularly important as students continue to develop their algebraic reasoning and spatial thinking.

Yu et al. (2009) discuss the idea of prototype and categorical thinking by describing an experiment where students are given visuals of three different rectangles, a vertical, long, and narrow one; a horizontal stout one, and a square. When asked to pick a rectangle, most students pick the horizontal one, as that is the one typically shown in geometry textbooks. This is also seen with students' understanding of other shapes, where a change in orientation often causes much confusion. Children develop their spatial reasoning through both play and focused mathematics instruction, and children's spatial skills strongly correlate to and predict future mathematics performance. As such, this is "an area that that demands greater attention in early childhood and elementary mathematics" (NCTM, 2020a, p. 117). These early experiences of translating between and within these representations are important for students' later understandings of geometric ideas taught at the secondary level which will now be discussed.

Middle and Secondary Geometry Experiences

A central goal of grade 6-8 geometry is to support students in developing a way to talk about properties of shapes, which is consistent with van Hiele's level 3 (Smith et al., 2017). Middle school geometry focuses on examining angles, transformations, congruency and similarity, and the Pythagorean theorem (as described in Nolan et al., 2016). As students transition from elementary to middle school, visual and physical representations should not fade away, but rather need to be developed alongside symbolic representations (Tripathi, 2008). The visual context of a geometry problem plays an integral role in the discovery of number patterns and algebraic expressions, and through the pattern recognition and counting skills developed at the elementary level, and the use of concrete manipulatives, students in middle school can move towards discovering basic geometric formulas (Beigie, 2011). While a focus of middle school geometry instruction is developing formulas, such as those for surface area and volume, these algebraic manipulations naturally lend themselves to connecting the concrete three-dimensional representation. It is important for students to cultivate this conceptual understanding so they can leverage these connections between and within representations and move beyond memorization and rote application (NCTM, 2020b).

High school geometry is often the first opportunity for a formal exploration of inductive and deductive reasoning and proofs. Additionally, high school geometry focuses on making sense of space and visualization as with using transformations, and determining relationships among measurements such as length, area, and volume. The four primary focuses of high school geometry include measurement; transformations; geometric arguments, reasoning, and proof; and solving applied problems and modeling in geometry (NCTM, 2018). "Geometry provides a bridge between many topics in mathematics. It connects functions to their representations, proportions to similar triangles, and triangles to trigonometry" (Nolan et al., 2016, p. 57). Even with these explicit algebraic connections, teachers need to make intentional efforts to connect the symbolic and algebraic to the physical and visual representations that are often brought forward through integrating geometric connections. For example, tasks that allow students to visualize two-and three-dimensional shapes and solids in multiple ways can support conceptual understanding of geometric concepts such as area, surface area, and volume (Ben-Haim et al., 1985; Ferrer et al., 2001) and can enable students to develop further and deeper meaning for these constructs (Smith et al., 2017). Safi and Desai (2017) suggest that teachers can use two- and three-dimensional manipulatives to emphasize connections between algebraic instances—such as multiplying polynomials—with the geometric representations related to the area accounted for through the product of algebraic expressions. Geometric and algebraic understandings and representations reinforce each other, and for students to gain a rich perspective, it is necessary to expose students to both.

In recent years, teachers and students have potentially greater access to new forms of dynamic representations, including open source and freely available dynamic geometry software, virtual manipulatives, and other apps that enable them to manipulate visualizations which was once not possible with the static paper-pencil methods (Hollebrands & Dove, 2011; Jackiw, 2001). As described by Battista's (2009) framework, briefly described in Table 1, such representations allow students to connect conceptual knowledge to dynamic pictorial representations, thereby providing rich opportunities for understanding and connecting geometric representations. Hollebrands (2003) recommends that teachers use dynamic geometry software to support students in gaining deeper understanding of transformational geometry concepts and the connections between transformations and functions. Dynamic software applications introduce students to mathematics that would have otherwise been out of reach and help students transfer metal images of concepts to visual interactive representations that can lead to more robust understanding (Dick & Hollebrands, 2011). Much of secondary mathematics focuses on formal and rigorous mathematical reasoning, and oftentimes there is a greater emphasis on algebraic or symbolic manipulation and logical deductions (Battista, 2017). While this emphasis is indeed necessary, it is equally important for students to be given experiences with other forms of representations (both static and dynamic) to build their initial conceptions of the topic. Such explorations allow for connecting multiple mathematical representations while providing affordances from each representation that can be leveraged in future mathematical explorations.

Concluding Remarks

This literature review synthesizes, organizes, and elaborates on existing literature relating to the teaching and learning of geometry at the PK-12 level in the U.S. and the role that mathematical representations can play in enriching the geometry experience for our students. Through this synthesis, it is evident that representations play a crucial role in the teaching and learning of geometry. By providing students access to opportunities to explore multiple mathematical representations, they are no longer limited by the strengths and weaknesses of one particular representation (Elia et al., 2007), and they are able to deepen their mathematical understanding while engaging in meaningful mathematical discourse (Lesh et al., 1987; NCTM, 2014). Yet, as NCTM (2020b) discusses, at the early childhood and elementary level "Geometry instruction, typically, does not move beyond shape names or definitions, only engaging in low-level thinking" (p. 119). As evidenced within the geometry curriculum, this is similar at the secondary level where algebraic and symbolic representations are greatly overemphasized (Knuth, 2000). As a result, students often experience a disconnect in transferring between and with representations because some representations, especially symbolic and visual, are included as end products rather than as starting points in reasoning and problem solving. "Children enter this world as emergent mathematicians, naturally curious, and trying to make sense of their mathematical environment" (NCTM, 2020a, p. 17). For our students to continue to see themselves as capable learners and doers of mathematics and experience the wonder, joy, and beauty of doing mathematics, it is important that PK-12 instruction provides them opportunities to see connections between mathematics and their daily lives (NCTM 2018, 2020a, 2020b).

Giving students such opportunities to engage in tasks that allow the use of multiple mathematical representations empowers teachers to create more equitable tasks as they afford a wider range of access to mathematical ideas (Boston et al., 2017). However, the use of multiple mathematical representations is often placed into the curriculum as an afterthought to help students who may be struggling to firmly understand the content. Van de Walle, Karp, and Bay-Williams (2019) describe understanding as existing along a continuum from instrumental to relational understanding, terms that were first introduced by Richard Skemp in 1976. While students may perform well academically in the moment, as teachers it is important to think *about* and *through* whether students are just remembering or whether they are thinking about the mathematics. Clements (2003) supports the idea that all students need to play with concrete objects and see visual representations before they are able to understand abstract topics. Students get little meaningful mathematics out of the traditional proof-based approach that is often used in the high school geometry curriculum. Some students may be able to remember and give an output in the given amount of time, but "if we look at the mathematics in the world and the mathematics used by mathematicians, we see a creative, visual, connected and living subject" (Boaler, 2016, p. 31). Geometry naturally lends itself to noticing and wondering about the world around us and provides an ideal platform to make these representational connections a reality. Consistent PK-12 geometrical learning experiences through high quality tasks rightfully affords students intentional opportunities to translate between and within multiple mathematical representations, empowering students to experience this wonder, joy, and beauty in mathematics (NCTM 2018, 2020a, 2020b). In this manner, students' experiences can be fueled by discovering connections between and within mathematical representations, while linking mathematical domains enriched by the wonders of geometry in our lives, communities, and cultures.

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